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Gravity from spontaneous Lorentz violation

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We investigate a class of theories involving a symmetric two-tensor field in Minkowski spacetime with a potential triggering spontaneous violation of Lorentz symmetry. The resulting massless Nambu-Goldstone modes are shown to obey the linearized Einstein equations in a fixed gauge. Imposing self-consistent coupling to the energy-momentum tensor constrains the potential for the Lorentz violation. The nonlinear theory generated from the self-consistent bootstrap is an alternative theory of gravity, containing kinetic and potential terms along with a matter coupling. At energies small compared to the Planck scale, the theory contains general relativity, with the Riemann-spacetime metric constructed as a combination of the two-tensor field and the Minkowski metric. At high energies, the structure of the theory is qualitatively different from general relativity. Observable effects can arise in suitable gravitational experiments.

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I. INTRODUCTION

The idea that physical Lorentz symmetry could be broken in a fundamental theory of nature has received much attention in recent years. One attractive mechanism is spontaneous Lorentz violation, in which an interaction drives an instability that triggers the development of nonzero vacuum values for one or more tensor fields [1]. Unlike explicit breaking, spontaneous Lorentz violation is compatible with conventional gravitational geometries [2], and it is therefore advantageous for model building. However, spontaneous violation of a continuous global symmetry comes with massless excitations, the Nambu-Goldstone (NG) modes [3]. Among the challenges facing attempts to construct realistic models with spontaneous Lorentz violation is accounting for the role of the corresponding NG modes and interpreting them phenomenologically.

Since the NG modes are intrinsically massless, they can generate long-range forces. One intriguing possibility is that they could reproduce one of the long-range forces known to exist in nature. For electrodynamics, for example, the Einstein-Maxwell equations in a fixed gauge naturally emerge from the NG sector of certain gravitationally coupled vector theories with spontaneous Lorentz violation known as bumblebee models [4,5]. For gravity itself, the gravitons can be interpreted as the NG modes from spontaneous Lorentz violation in several ways. As fundamental field excitations, gravitons can be identified with the NG modes of a symmetric two-tensor field $C^{\mu\nu}$ in a theory with a potential inducing spontaneous Lorentz violation, which generates the linearized Einstein equations in a fixed gauge [6]. Alternatively, gravitons as composite objects can be understood as the NG modes of spontaneous Lorentz violation arising from self-interactions of vectors [7], fermions [8], or scalars [9], following related ideas for composite photons [10]. Other interpretations of the NG modes include a new spin-dependent interaction [11] and various new spin-independent forces [12]. For certain theories in Riemann-Cartan spacetimes, the NG modes can instead be absorbed into the spin connection via the Lorentz-Higgs effect [5].

In the present work, we investigate the possibility that the full nonlinear structure of general relativity can be recovered from an alternative theory of gravity with spontaneous Lorentz violation in which the gravitons are fundamental excitations identified with the NG modes. General relativity has the interesting feature that it can be reconstructed uniquely from massless spin-2 fields by requiring consistent self-coupling to the energy-momentum tensor [13–17]. For example, the linearized theory describing gravitational waves via a symmetric two-tensor $h^{\mu\nu}$ propagating in a spacetime with Minkowski metric $\eta_{\mu\nu}$ contains sufficient information to reconstruct the full nonlinearity of general relativity when self-consistency is imposed. Here, we demonstrate that applying this bootstrap method to a linearized theory with a symmetric two-tensor field $C^{\mu\nu}$ and a potential $V(C^{\mu\nu}, \eta_{\mu\nu})$ inducing spontaneous Lorentz violation yields an alternative theory of gravity, which we call cardinal gravity [18]. The coupling of the cardinal field to the matter sector is derived, and constraints from existing experiments are considered. We show that the action of cardinal gravity corresponds to the Einstein-Hilbert action at energies small compared to the Planck scale. However, the structure of the theory at high energies is qualitatively different from that of general relativity. Our results indicate that cardinal gravity is a viable alternative theory of gravity exhibiting some intriguing features in extreme gravitational environments.

We begin this work in Sec. II by presenting the linearized cardinal theory and a discussion of its correspondence to linearized general relativity. Section III reviews the bootstrap procedure for general relativity and obtains some generic results. For general relativity, the bootstrap procedure yields a unique answer even if a potential for $h^{\mu\nu}$ is allowed [16]. In the context of spontaneous Lorentz violation, the phase transition circumvents this uniqueness. However, the nontrivial integrability conditions required for implementing the bootstrap constrain the form of the potential V. In Sec. IV, we obtain differential equations expressing the integrability conditions and derive acceptable potentials V. This section also applies the bootstrap to yield the full cardinal gravity. Certain aspects of the extrema of the potential are considered, and alternative bootstrap procedures are discussed. The coupling of the cardinal field to the matter sector and some experimental implications are studied in Sec. V. A summary of the results and a discussion of their broader implications is provided in Sec. VI. Throughout this work, we use the conventions of Ref. [2].

II. LINEARIZED ANALYSIS

In this section, the linear cardinal theory is defined and investigated. We show that its NG sector is equivalent to conventional linearized gravity in a special gauge.

A. Linear cardinal theory

Consider first the action for the symmetric two-tensor cardinal field $C^{\mu\nu}$ defined in a background spacetime. For definiteness and simplicity, we take the background to be Minkowski spacetime with metric $\eta_{\mu\nu}$, although a more general background could be countenanced and treated with similar methods. We suppose the kinetic term in the action is quadratic in $C^{\mu\nu}$, so the derivative operators in the equation of motion are linear in $C^{\mu\nu}$. The NG excitations of $C^{\mu\nu}$ subsequently play the role of the metric fluctuation in a linearized theory of gravity. The action is assumed to generate spontaneous violation of Lorentz symmetry through a potential $V(C^{\mu\nu}, \eta_{\mu\nu})$.

1. Basics

The Lagrange density for the linear cardinal theory is taken to be

$$\mathcal{L}_C = \tfrac{1}{2} C^{\mu\nu} K_{\mu\nu\alpha\beta} C^{\alpha\beta} - V(C^{\mu\nu}, \eta_{\mu\nu}). \qquad (1)$$

Here, $K_{\mu\nu\alpha\beta}$ is the usual quadratic kinetic operator for a massless spin-2 field. Allowing for an arbitrary scaling parameter κ to be chosen later, $K_{\mu\nu\alpha\beta}$ can be written in Cartesian coordinates as

$$\begin{aligned} K_{\mu\nu\alpha\beta} = \tfrac{1}{2}\kappa\Big[\Big(-\eta_{\mu\nu}\eta_{\alpha\beta} + \tfrac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta} + \tfrac{1}{2}\eta_{\mu\beta}\eta_{\nu\alpha}\Big)\partial^\lambda\partial_\lambda \\ + \eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu \\ - \tfrac{1}{2}\eta_{\mu\alpha}\partial_\nu\partial_\beta - \tfrac{1}{2}\eta_{\nu\alpha}\partial_\mu\partial_\beta \\ - \tfrac{1}{2}\eta_{\mu\beta}\partial_\nu\partial_\alpha - \tfrac{1}{2}\eta_{\nu\beta}\partial_\mu\partial_\alpha\Big], \end{aligned} \qquad (2)$$

where $\eta_{\mu\nu}$ is the Minkowski metric with diagonal entries $(-1, 1, 1, 1)$ as the only nonzero components. As usual, in other coordinate systems the Minkowski metric takes different forms and covariant derivatives must be used. The equations of motion obtained by varying Eq. (1) with respect to $C^{\mu\nu}$ are

$$K_{\mu\nu\alpha\beta} C^{\alpha\beta} - \frac{\delta V}{\delta C^{\mu\nu}} = 0. \qquad (3)$$

The theory (1) has various symmetries. It is invariant under translations and under global Lorentz transformations. For infinitesimal parameters $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, the latter take the form

$$C^{\mu\nu} \to C^{\mu\nu} + \epsilon^\mu{}_\alpha C^{\alpha\nu} + \epsilon^\nu{}_\alpha C^{\alpha\mu}, \qquad \eta_{\mu\nu} \to \eta_{\mu\nu}. \qquad (4)$$

There are also local spacetime symmetries, including invariance under local Lorentz transformations on the tangent space at each point and invariance under diffeomorphisms of the Minkowski spacetime. These local symmetries play a subsidiary role in the present context.

In addition to the spacetime symmetries, the form of the kinetic operator (2) ensures that the kinetic term is by itself invariant under gauge transformations of $C^{\mu\nu}$ alone,

$$C^{\mu\nu} \to C^{\mu\nu} - \partial^\mu\Lambda^\nu - \partial^\nu\Lambda^\mu, \qquad \eta_{\mu\nu} \to \eta_{\mu\nu}. \qquad (5)$$

However, one or more of these four gauge symmetries may be explicitly broken by the potential V, so the Lagrange density (1) contains between zero and four gauge degrees of freedom depending on the choice of V. Since $C^{\mu\nu}$ has ten independent components, it follows that there are between six and ten physical or auxiliary fields.

The potential V for the theory (1) is a scalar function of the cardinal field $C^{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$. The only scalars that can be formed from these two objects involve traces of products of the combination $C^{\mu\alpha}\eta_{\alpha\nu}$. The scalar X_m with m such products has the form

$$X_m = \mathrm{tr}[(C\eta)^m]. \qquad (6)$$

Here, we have introduced a convenient matrix notation $(C\eta)^\mu{}_\nu \equiv C^{\mu\alpha}\eta_{\alpha\nu}$. Since $C\eta$ is a symmetric 4×4 matrix, there are at most four independent scalars X_m, so we can restrict attention to the cases $X_m = 1, 2, 3, 4$. It follows that the potential V can be written as

$$V = V(X_1, X_2, X_3, X_4) \qquad (7)$$

without loss of generality. For definiteness, V is assumed to be positive everywhere except at its absolute minimum, which is taken to be zero.

Under the gauge transformation (5), each scalar X_m transforms nontrivially and therefore explicitly breaks one symmetry. For simplicity in what follows, we assume the potential V depends on all four independent scalars X_m, so the gauge symmetry (5) is completely broken for generic field configurations. With this assumption, the theory describes ten physical or auxiliary fields and zero gauge fields. This assumption could be relaxed, but the resulting discussion would involve additional gauge-fixing considerations.

The potential V is taken to have a minimum in which $C^{\mu\nu}$ attains a nonzero vacuum value

$$\langle C^{\mu\nu}\rangle \equiv c^{\mu\nu}. \tag{8}$$

In this minimum, the scalars X_m have vacuum values

$$\langle X_m\rangle \equiv x_m = \mathrm{tr}[(c\eta)^m]. \tag{9}$$

These vacuum values spontaneously break particle Lorentz symmetry, but they leave unaffected the structure of observer Lorentz and general coordinate transformations, which amount to coordinate choices. To avoid complications with soliton-type solutions, we also suppose $c^{\mu\nu}$ is constant,

$$\partial_\alpha c^{\mu\nu} = 0 \tag{10}$$

in Cartesian coordinates.

Given a vacuum value $c^{\mu\nu}$, the freedom of coordinate choice can be used to adopt a canonical form. For definiteness and simplicity, we assume in what follows that the matrix $(c\eta)^\mu{}_\nu \equiv c^{\mu\alpha}\eta_{\alpha\nu}$ has four inequivalent nonzero real eigenvalues. This implies, for example, invertibility and the existence of one timelike and three spacelike eigenvectors. It also implies that all six Lorentz transformations are spontaneously broken. The consequences of other possible choices may differ from the discussion below and would be interesting to explore, but they lie beyond our present scope.

2. Nambu-Goldstone and massive modes

The physical degrees of freedom contained in the cardinal field $C^{\mu\nu}$ can be taken as fluctuations about the vacuum value $c^{\mu\nu}$. We write

$$C^{\mu\nu} = c^{\mu\nu} + \tilde{C}^{\mu\nu}. \tag{11}$$

The fluctuation field $\tilde{C}^{\mu\nu}$ is symmetric and has ten independent components, which include both the NG modes and the massive modes in the theory.

To identify the NG modes, we can make virtual infinitesimal symmetry transformations using the broken generators acting on field vacuum values, and then promote the corresponding parameters to field excitations. An infinitesimal Lorentz transformation with parameters $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ yields

$$\langle C^{\mu\nu}\rangle \to c^{\mu\nu} + \epsilon^\mu{}_\alpha c^{\alpha\nu} + \epsilon^\nu{}_\alpha c^{\alpha\mu}. \tag{12}$$

Since there are six Lorentz transformations (three rotations and three boosts), there could in principle be up to six Lorentz NG modes, corresponding to the promotion of the six parameters $\epsilon_{\mu\nu}$ to fields $\mathcal{E}_{\mu\nu} = -\mathcal{E}_{\nu\mu}$ [5,19]. For $c^{\mu\nu}$ satisfying our assumed conditions, the maximal set of six NG modes appears. In general, the NG modes in $\tilde{C}^{\mu\nu}$ are contained in the fluctuations $N^{\mu\nu}$ defined by

$$\tilde{C}^{\mu\nu} \supset N^{\mu\nu} = \mathcal{E}^\mu{}_\alpha c^{\alpha\nu} + \mathcal{E}^\nu{}_\alpha c^{\alpha\mu} \equiv O^{\mu\nu\alpha\beta}\mathcal{E}_{\alpha\beta}, \tag{13}$$

where

$$O^{\mu\nu\alpha\beta} = \tfrac{1}{2}(\eta^{\mu\alpha}c^{\nu\beta} + \eta^{\nu\alpha}c^{\mu\beta} - \eta^{\mu\beta}c^{\nu\alpha} - \eta^{\nu\beta}c^{\mu\alpha}). \tag{14}$$

Since there are six independent fields in $\mathcal{E}_{\mu\nu}$, the ten symmetric components of $N^{\mu\nu}$ must obey four identities. For $c^{\mu\nu}$ satisfying our assumed conditions, we find these identities can be expressed as

$$\mathrm{tr}[N\eta(c\eta)^j] = 0, \tag{15}$$

with $j = 0, 1, 2, 3$.

In addition to the six NG modes in the field $N^{\mu\nu}$, the fluctuation $\tilde{C}^{\mu\nu}$ includes four massive modes. These are contained in the field $M^{\mu\nu}$ given by

$$M^{\mu\nu} = \tilde{C}^{\mu\nu} - N^{\mu\nu}, \tag{16}$$

subject to a suitable orthogonality condition. The symmetric field $M^{\mu\nu}$ has ten components but only four independent degrees of freedom, which we denote here by m_j, $j = 0, 1, 2, 3$. For some purposes, it is convenient to expand $M^{\mu\nu}$ as

$$M^{\mu\nu} = m_0\eta^{\mu\nu} + m_1 c^{\mu\nu} + m_2(c\eta c)^{\mu\nu} + m_3(c\eta c\eta c)^{\mu\nu}. \tag{17}$$

The fields $N^{\mu\nu}$ and $M^{\mu\nu}$ obey identities expressing a kind of orthogonality:

$$\mathrm{tr}[N\eta(M\eta)^j] = 0, \tag{18}$$

with $j = 0, 1, 2, 3$. More generally, we find

$$\mathrm{tr}[N\eta F(c\eta, M\eta)] = 0, \tag{19}$$

where $F(c\eta, M\eta)$ is an arbitrary matrix polynomial in $c\eta$ and $M\eta$.

With the expansion (17), the fluctuation $\tilde{C}^{\mu\nu}$ can be written

$$\tilde{C}^{\mu\nu} = N^{\mu\nu} + \sum_{j=0}^{3} m_j[(c\eta)^j]^\mu{}_\alpha\eta^{\alpha\nu}. \tag{20}$$

Using this equation, the four massive modes m_j can be expressed in terms of $\tilde{C}^{\mu\nu}$. Multiplying by $[\eta(c\eta)^k]_{\mu\nu}$ with $k = 0, 1, 2, 3$ and applying the identities (15) yields the four equations

$$\begin{pmatrix} 4 & \mathrm{tr}[c\eta] & \mathrm{tr}[(c\eta)^2] & \mathrm{tr}[(c\eta)^3] \\ \mathrm{tr}[c\eta] & \mathrm{tr}[(c\eta)^2] & \mathrm{tr}[(c\eta)^3] & \mathrm{tr}[(c\eta)^4] \\ \mathrm{tr}[(c\eta)^2] & \mathrm{tr}[(c\eta)^3] & \mathrm{tr}[(c\eta)^4] & \mathrm{tr}[(c\eta)^5] \\ \mathrm{tr}[(c\eta)^3] & \mathrm{tr}[(c\eta)^4] & \mathrm{tr}[(c\eta)^5] & \mathrm{tr}[(c\eta)^6] \end{pmatrix}\begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} \mathrm{tr}[\tilde{C}\eta] \\ \mathrm{tr}[\tilde{C}\eta(c\eta)] \\ \mathrm{tr}[\tilde{C}\eta(c\eta)^2] \\ \mathrm{tr}[\tilde{C}\eta(c\eta)^3] \end{pmatrix}. \tag{21}$$

The traces $\mathrm{tr}[(c\eta)^p]$ with $p = 5$, 6 can be rewritten in terms of $\mathrm{tr}[(c\eta)^m]$ with $m = 1$, 2, 3, 4 using the Hamilton-Cayley theorem. In terms of the eigenvalues c_j of the matrix $c\eta$, the determinant of the 4×4 matrix $\mathcal{O}$ on the left-hand side takes the form

$$\det[\mathcal{O}] = \prod_{\substack{j,k=0 \\ j<k}}^{3} (c_j - c_k)^2. \tag{22}$$

For the matrix $c\eta$ satisfying our assumed conditions, it follows that Eq. (21) can be inverted to give explicit expressions for each of the four massive modes m_j in terms of $\tilde{C}^{\mu\nu}$. These somewhat lengthy expressions involve the four field traces $\mathrm{tr}[\tilde{C}\eta(c\eta)^j]$ with $j = 0$, 1, 2, 3 and the four quantities $\mathrm{tr}[(c\eta)^m]$ with $m = 1, 2, 3, 4$. Their explicit forms are unnecessary in the discussion that follows, so we omit them here.

The above considerations reveal that the decomposition of the cardinal field $C^{\mu\nu}$ in terms of NG and massive modes is

$$C^{\mu\nu} = c^{\mu\nu} + N^{\mu\nu} + M^{\mu\nu}. \tag{23}$$

The potential V can therefore be viewed as a function of $N^{\mu\nu}$ and $M^{\mu\nu}$ with constraints added to restrict these fields to their independent degrees of freedom, or equivalently as a function of the Lorentz NG modes $\mathcal{E}_{\mu\nu}$ and the massive modes m_j:

$$V(C^{\mu\nu}, \eta_{\mu\nu}) = V(c^{\mu\nu}, \mathcal{E}_{\mu\nu}, m_0, m_1, m_2, m_3, \eta_{\mu\nu}). \tag{24}$$

To investigate the correspondence of the linear cardinal theory (1) to linearized general relativity, it is useful to restrict attention to the pure NG sector. This can be achieved by considering the limit of infinite mass for the fields m_j. Alternatively, the potential V can be replaced with the Lagrange-multiplier limit V_λ given by

$$V_\lambda = \sum_{m=1}^{4} \lambda_m (X_m - x_m), \tag{25}$$

where the quantities λ_m are four Lagrange-multiplier fields. This potential freezes all fluctuations of $C^{\mu\nu}$ away from the potential minimum. In this limit, the independent degrees of freedom in the field fluctuations $\tilde{C}^{\mu\nu}$ are therefore restricted to the NG modes $\mathcal{E}_{\mu\nu}$ or, equivalently, $\tilde{C}^{\mu\nu} \rightarrow N^{\mu\nu}$ subject to the constraints (15). If desired, the on-shell values of λ_j can be set to zero by a suitable choice of initial conditions. Equivalent results could be obtained via an alternative Lagrange density involving a potential V with quadratic Lagrange-multiplier terms instead [19]. In any event, if the graviton is to be identified with the Lorentz NG modes in the theory (1), it follows that the field $N^{\mu\nu}$ must be the candidate graviton field.

3. Equations of motion for NG modes

The behavior of the candidate graviton field $N^{\mu\nu}$ is determined by its equations of motion. In the pure NG sector with vanishing Lagrange multipliers, the theory (1) with the potential (25) is equivalent to an effective Lagrange density $\mathcal{L}_{\mathrm{NG}}$ for the independent degrees of freedom, which are the Lorentz NG modes $\mathcal{E}_{\mu\nu}$. We can therefore write

$$\mathcal{L}_{\mathrm{NG}} = \tfrac{1}{2} O^{\mu\nu\rho\sigma} \mathcal{E}_{\rho\sigma} K_{\mu\nu\alpha\beta} O^{\alpha\beta\gamma\delta} \mathcal{E}_{\gamma\delta}. \tag{26}$$

Varying $\mathcal{L}_{\mathrm{NG}}$ with respect to the independent degrees of freedom $\mathcal{E}_{\mu\nu}$ yields the six equations of motion

$$O^{\mu\nu\rho\sigma} K_{\mu\nu\alpha\beta} O^{\alpha\beta\gamma\delta} \mathcal{E}_{\gamma\delta} = 0. \tag{27}$$

These can equivalently be written as

$$O^{\mu\nu\rho\sigma} K_{\mu\nu\alpha\beta} N^{\alpha\beta} = 0, \tag{28}$$

where the constraints (15) are understood.

To solve these equations we can use Fourier decomposition, transforming to momentum space with 4-momentum k_μ. It is convenient to introduce the scalars $K_{m,n}$ and K_m, defined by the matrix equations

$$K_{m,n} \equiv k(c\eta)^m N \eta (c\eta)^n k, \qquad K_m \equiv k(c\eta)^m k. \tag{29}$$

Note that $K_{m,n} = K_{n,m}$ by virtue of the symmetry of $N^{\mu\nu}$. Contraction of the equations of motion (28) with $k(c\eta)^m$ yields the following results, equivalent in content to the original equations of motion:

$$k^2 K_{m+1,n} + K_m K_{0,n+1} + K_{n+1} K_{m,0} - k^2 K_{m,n+1} - K_n K_{m+1,0} - K_{m+1} K_{0,n} = 0. \tag{30}$$

These expressions are solved by the on-shell condition $k^2 = 0$ and the constraint $k_\mu N^{\mu\nu} = 0$. We have verified that no physical off-shell solutions exist. The on-shell solutions are modes obeying the usual massless wave equation,

$$\partial_\lambda \partial^\lambda N^{\mu\nu} = 0, \tag{31}$$

subject to the harmonic condition

$$\partial_\mu N^{\mu\nu} = 0. \tag{32}$$

The latter imposes four constraints on the six independent degrees of freedom in $N^{\mu\nu}$.

We thus see that only two combinations of the six massless Lorentz NG modes $\mathcal{E}_{\mu\nu}$ propagate as physical on-shell fields. The other four NG modes are auxiliary. With the full potential V replaced by the Lagrange-multiplier limit V_λ, the four Lagrange multipliers can be viewed as playing a role analogous to that of the four frozen massive modes m_j.

B. Correspondence to linearized general relativity

In this subsection, we show the correspondence between the restriction of the linear cardinal theory to the NG sector and the usual weak-field limit of general relativity describing a massless spin-2 graviton field $h_{\mu\nu}$ propagating in a background Minkowski spacetime.

Consider the Lagrange density for a free symmetric massless spin-2 field $h_{\mu\nu}$, which is of the form (1) with $C^{\mu\nu}$ replaced by $h^{\mu\nu}$ and the potential V set to zero:

$$\mathcal{L}_h = \tfrac{1}{2} h^{\mu\nu} K_{\mu\nu\alpha\beta} h^{\alpha\beta}. \tag{33}$$

The definition of $K_{\mu\nu\alpha\beta}$ in Eq. (2) implies

$$K_{\mu\nu\alpha\beta} h^{\alpha\beta} \equiv -\kappa G^L_{\mu\nu}, \tag{34}$$

where $G^L_{\mu\nu}$ is the Einstein tensor linearized in $h^{\mu\nu}$. At this stage, the value of κ can be fixed by requiring a match to the conventional normalization of the linearized action for general relativity in the presence of a matter coupling given by

$$\mathcal{L}^L_T = \tfrac{1}{2} h^{\mu\nu} T_{M\mu\nu}, \tag{35}$$

where $T_{M\mu\nu}$ is the matter energy-momentum tensor. This match fixes κ to be

$$\kappa = \frac{1}{16\pi G_N}, \tag{36}$$

where G_N is the Newton gravitational constant.

A priori, $h^{\mu\nu}$ has ten degrees of freedom. However, the theory is invariant under the four gauge transformations

$$h^{\mu\nu} \to h^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu, \tag{37}$$

so four gauge-fixing conditions can be imposed on $h^{\mu\nu}$. Numerous choices of gauge appear in the literature. For free wave propagation, a common choice is transverse-traceless gauge, which imposes

$$n_\mu h^{\mu\nu} = 0, \qquad h \equiv h^\mu{}_\mu = 0, \tag{38}$$

for a unit timelike vector n_μ. For suitable initial conditions, the harmonic condition

$$\partial_\mu h^{\mu\nu} = 0 \tag{39}$$

then follows from the equations of motion. However, this gauge is not the only possible choice. Here, we demonstrate the existence of an alternative gauge condition on $h^{\mu\nu}$ that yields directly a match to the NG effective Lagrange density (26).

The conditions fixing this alternative "cardinal" gauge at linear order in $h^{\mu\nu}$ are

$$\mathrm{tr}[h\eta(c\eta)^j] = 0, \tag{40}$$

where $j = 0, 1, 2, 3$. In this expression, $(c\eta)^\mu{}_\nu \equiv c^{\mu\alpha}\eta_{\alpha\nu}$ is a constant matrix assumed to have four inequivalent nonzero real eigenvalues, which we denote by c_j, $j = 0, 1, 2, 3$. This assumption ensures the four conditions (40) are independent. For the present purpose of matching to the linear cardinal theory (1), the quantity $c^{\mu\nu}$ is to be identified with the vacuum value of $C^{\mu\nu}$ in Eq. (8), so we denote it by the same symbol.

To show that the conditions (40) are indeed a choice of gauge, we can consider an arbitrary initial field $h'^{\mu\nu}$ and seek quantities ξ^μ such that a gauge transformation of the form (37) generates the desired field $h^{\mu\nu}$ satisfying (40). In momentum space, the gauge transformation (37) takes the form

$$h^{\mu\nu} = h'^{\mu\nu} - ik^\mu\xi^\nu - ik^\nu\xi^\mu. \tag{41}$$

The requirements on ξ^μ become

$$\begin{aligned} ik_\mu\xi^\mu &= \tfrac{1}{2}\mathrm{tr}[\eta h'], \\ ik_\alpha(c\eta)^\alpha{}_\mu\xi^\mu &= \tfrac{1}{2}\mathrm{tr}[\eta h'\eta c], \\ ik_\alpha[(c\eta)^2]^\alpha{}_\mu\xi^\mu &= \tfrac{1}{2}\mathrm{tr}[\eta h'(\eta c)^2], \\ ik_\alpha[(c\eta)^3]^\alpha{}_\mu\xi^\mu &= \tfrac{1}{2}\mathrm{tr}[\eta h'(\eta c)^3]. \end{aligned} \tag{42}$$

This represents a set of four equations for the four unknowns ξ^μ, which can be regarded as a matrix equation. The set has a unique solution if the 4×4 matrix generated by the coefficients of ξ^μ is invertible. Then, the four 4-vectors k_μ, $k_\alpha(c\eta)^\alpha{}_\mu$, $k_\alpha[(c\eta)^2]^\alpha{}_\mu$, $k_\alpha[(c\eta)^3]^\alpha{}_\mu$ are linearly independent, and so

$$\epsilon_{\mu\nu\rho\sigma}k^\mu k_\alpha c^{\alpha\nu}k_\beta(c\eta c)^{\beta\rho}k_\gamma(c\eta c\eta c)^{\gamma\sigma} \neq 0. \tag{43}$$

Expanding the 4-vector k in terms of the eigenvectors $e^{(a)}$ of the matrix $c\eta$ shows that this condition is indeed satisfied for generic k, for which all components $k^{(a)} = k\cdot e^{(a)}$ are nonzero. It follows that the cardinal gauge (40) can be attained everywhere in conventional linearized general relativity, except for special k at which additional gauge fixing is required. This remnant gauge freedom is analogous to that of axial gauge in electrodynamics [20]. Similarly, in the context of spontaneous Lorentz violation, the linearized potential for the vector field in certain bumblebee models generates an NG-sector axial constraint with a related remnant gauge freedom [5,19]. For simplicity in what follows, we consider the case of generic k.

Once the cardinal gauge (40) is imposed, the harmonic condition (32) follows from the equations of motion. The latter are found from the Lagrange density (33) to be

$$K_{\mu\nu\alpha\beta}h^{\alpha\beta} \equiv -\kappa G^L_{\mu\nu} = 0. \tag{44}$$

Contracting these equations in turn with $\eta^{\mu\nu}$, $c^{\mu\nu}$, $(c\eta c)^{\mu\nu}$, and $(c\eta c\eta c)^{\mu\nu}$ yields in momentum space the four conditions

$$\begin{aligned} k_\mu h^{\mu\nu}k_\nu &= 0, \\ k_\alpha c^\alpha{}_\mu h^{\mu\nu}k_\nu &= 0, \\ k_\alpha c^\alpha{}_\beta c^\beta{}_\mu h^{\mu\nu}k_\nu &= 0, \\ k_\alpha c^\alpha{}_\beta c^\beta{}_\gamma c^\gamma{}_\mu h^{\mu\nu}k_\nu &= 0. \end{aligned} \tag{45}$$

Collecting the coefficients of $h^{\mu\nu}k_\nu$ gives a 4×4 matrix that is invertible when Eq. (43) is satisfied, which is the case under the present assumptions. It follows that $h^{\mu\nu}k_\nu = 0$, and hence that the gauge choice (40) obeys the harmonic condition (32). The equations of motion then reduce to

$$\partial^\lambda\partial_\lambda h^{\mu\nu} = 0, \tag{46}$$

and they describe the usual two graviton degrees of freedom propagating as massless spin-2 waves.

We now have all the ingredients in hand to verify the equivalence between the theory (33) for a propagating spin-2 field $h^{\mu\nu}$ and the theory (26) for the NG sector of the cardinal model. Starting with the former, we can impose the four cardinal gauge conditions (40) on the ten independent graviton components $h^{\mu\nu}$. The equations of motion (44) then imply the harmonic condition (32), which leaves two degrees of freedom that propagate as conventional massless modes. These results are paralleled in the theory (26) for the NG sector of the cardinal model. The field $N^{\mu\nu}$ containing the Lorentz NG modes $\mathcal{E}_{\mu\nu}$ is subject to the constraints (15), so $N^{\mu\nu}$ matches the graviton $h^{\mu\nu}$ in cardinal gauge,

$$h^{\mu\nu} \leftrightarrow N^{\mu\nu}. \tag{47}$$

The harmonic condition holds for both $N^{\mu\nu}$ and $h^{\mu\nu}$. The equations of motion (28) for the Lorentz NG modes $\mathcal{E}_{\mu\nu}$ can be matched directly to the equations of motion (44) for the graviton $h^{\mu\nu}$ by multiplying the latter with $O^{\mu\nu\rho\sigma}$.

Evidently, the cardinal and graviton theories are in direct correspondence, even though their gauge structures differ. The presence of the potential in the linear cardinal theory excludes the gauge symmetry of the graviton theory, but the gauge freedom of the latter means that only six of the ten components of $h^{\mu\nu}$ are physical or auxiliary, thereby matching the six Lorentz modes $\mathcal{E}_{\mu\nu}$ in the NG sector of the cardinal theory. Note also that the gauge freedom of the graviton theory could be fixed to cardinal gauge in a standard way, by adding suitable gauge-fixing terms to the Lagrange density. The parallel in the cardinal theory would be the presence of Lagrange multipliers for the constraints (15).

III. BOOTSTRAP PROCEDURE

This section considers some generic features of the bootstrap procedure for self-consistent coupling to the energy-momentum tensor. We summarize the Deser version [14] of the bootstrap for obtaining general relativity from the linear graviton theory (33), and we present some generic results that are useful for the subsequent analysis.

A. Bootstrap for general relativity

The analysis takes advantage of the first-order Palatini form [21] of the nonlinear Einstein-Hilbert action of general relativity, which can be written as

$$S_{\rm GR} = \int d^4x\kappa\mathfrak{g}^{\mu\nu}R_{\mu\nu}(\Gamma). \tag{48}$$

Here, $\mathfrak{g}^{\mu\nu}$ is the tensor density of weight one defined in terms of the usual reciprocal metric $g^{\mu\nu}$ as

$$\mathfrak{g}^{\mu\nu} \equiv \sqrt{|g|}g^{\mu\nu}. \tag{49}$$

Its inverse is a tensor density of weight negative one, which we define as

$$\mathfrak{g}_{\mu\nu} \equiv \frac{1}{\sqrt{|g|}}g_{\mu\nu}. \tag{50}$$

Also,

$$\begin{aligned} R_{\mu\nu}(\Gamma) = {}& \partial_\alpha\Gamma^\alpha{}_{\mu\nu} - \tfrac{1}{2}\partial_\mu\Gamma^\alpha{}_{\nu\alpha} - \tfrac{1}{2}\partial_\nu\Gamma^\alpha{}_{\mu\alpha} \\ &+ \Gamma^\beta{}_{\beta\alpha}\Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha} \end{aligned} \tag{51}$$

is the curvature tensor for the connection $\Gamma^\alpha{}_{\mu\nu}$. In this approach, both $\mathfrak{g}^{\mu\nu}$ and $\Gamma^\alpha{}_{\mu\nu}$ are viewed as independent fields at the level of the action, and the identification of $\Gamma^\alpha{}_{\mu\nu}$ with the Christoffel symbols arises on shell by solving the equations of motion.

In what follows, we define the fluctuation $\mathfrak{h}^{\mu\nu}$ of $\mathfrak{g}^{\mu\nu}$ about the Minkowski background $\eta^{\mu\nu}$ as

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \mathfrak{h}^{\mu\nu}. \tag{52}$$

Note the use of contravariant indices in this definition. Also, note that $\mathfrak{h}^{\mu\nu}$ can be identified at linear order with the usual trace-corrected field $\bar{h}^{\mu\nu}$:

$$\mathfrak{h}^{\mu\nu} \approx -\bar{h}^{\mu\nu} \equiv -h^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}h. \tag{53}$$

Given the linear graviton theory (33), the nonlinear Einstein-Hilbert action can be derived by adding a coupling to the energy-momentum tensor $T_{\mu\nu}$ and requiring its conservation be consistent order by order [13]. Deser has shown that this bootstrap procedure can be performed in a single elegant step [14].

The starting point of the derivation is to note that the equations of motion (44) for $h^{\mu\nu}$, obtained in the previous section from the second-order Lagrange density (33), also follow from the linearized version of the first-order action

(48). The latter becomes

$$S^L_{\rm GR} = \int d^4x \mathcal{L}^L_{\rm GR},$$
$$\mathcal{L}^L_{\rm GR} = \kappa[\mathfrak{h}^{\mu\nu}(\partial_\alpha\Gamma^\alpha{}_{\mu\nu} - \partial_\nu\Gamma^\alpha{}_{\mu\alpha}) + \eta^{\mu\nu}(\Gamma^\beta{}_{\beta\alpha}\Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha})], \tag{54}$$

with $\mathfrak{h}^{\mu\nu}$ and $\Gamma^\alpha{}_{\mu\nu}$ viewed as independent fields. Variation of $S^L_{\rm GR}$ with respect to these fields yields two sets of equations of motion. These fix $\Gamma^\alpha{}_{\mu\nu}$ as the usual linearized Christoffel symbols, and they imply the linear equations of motion (44) for $h^{\mu\nu}$ obtained from the second-order Lagrange density (33).

The prescription for the bootstrap procedure is to require that the energy-momentum tensor $T_{\mu\nu}$ obtained from the action (54) is coupled as a source in a self-consistent manner. It turns out to be most convenient to work with the trace-reversed energy-momentum tensor $\tau_{\mu\nu}$, which in the linear limit is related to $T_{\mu\nu}$ by

$$\tau_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T^\alpha{}_\alpha. \tag{55}$$

For a given Lagrange density $\mathcal{L}$ in Minkowski spacetime with metric $\eta_{\mu\nu}$, the tensor $\tau_{\mu\nu}$ can be calculated via the Rosenfeld method [22]. The procedure involves promoting the Minkowski metric $\eta^{\mu\nu}$ to an auxiliary weight-one metric density $\psi^{\mu\nu}$ and the partial derivative ∂_μ to the covariant derivative D_μ formed using $\psi^{\mu\nu}$, so that $\mathcal{L}$ becomes covariant in the auxiliary spacetime. The trace-reversed energy-momentum tensor $\tau_{\mu\nu}$ is then found from the expression

$$-\frac{1}{2}\tau_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\psi^{\mu\nu}}\bigg|_{\psi\to\eta}. \tag{56}$$

For the linear theory with Lagrange density $\mathcal{L}^L_{\rm GR}$, this yields

$$-\tfrac{1}{2}\tau_{\mathfrak{h}\mu\nu} = \kappa(\Gamma^\beta{}_{\beta\alpha}\Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha}) + \kappa\sigma_{\mu\nu}(\mathfrak{h},\Gamma), \tag{57}$$

where $\sigma_{\mu\nu}$ is a total-derivative term given by

$$\sigma^{\mu\nu}(\mathfrak{h},\Gamma) = -\tfrac{1}{2}\partial_\gamma[\mathfrak{h}^{\mu\gamma}\Gamma^\rho{}_\rho{}^\nu + \mathfrak{h}^{\nu\gamma}\Gamma^\rho{}_\rho{}^\mu - \mathfrak{h}^{\mu\nu}\Gamma^\rho{}_\rho{}^\gamma + \mathfrak{h}^{\mu\rho}(\Gamma^\nu{}_\rho{}^\gamma - \Gamma^\gamma{}_\rho{}^\nu) + \mathfrak{h}^{\nu\rho}(\Gamma^\mu{}_\rho{}^\gamma - \Gamma^{\gamma\mu}{}_\rho) - \mathfrak{h}^{\gamma\rho}(\Gamma^\mu{}_\rho{}^\nu - \Gamma^\nu{}_\rho{}^\mu) + \eta^{\mu\nu}(\tfrac{1}{2}{\rm tr}[\mathfrak{h}\eta]\Gamma^\sigma{}_\sigma{}^\gamma - \mathfrak{h}^{\rho\sigma}\Gamma_{\rho\sigma}{}^\gamma)]. \tag{58}$$

On shell, $\sigma_{\mu\nu}$ can be expressed more elegantly as

$$\sigma_{\mu\nu} = R^L_{\mu\nu}(\Gamma) - R^L_{\mu\nu}(\Gamma^L), \tag{59}$$

where $R^L_{\mu\nu}(\Gamma)$ is the linear part of the Ricci curvature

$$R^L_{\mu\nu}(\Gamma) = \partial_\alpha\Gamma^\alpha{}_{\mu\nu} - \tfrac{1}{2}\partial_\mu\Gamma^\alpha{}_{\nu\alpha} - \tfrac{1}{2}\partial_\nu\Gamma^\alpha{}_{\mu\alpha}, \tag{60}$$

and Γ^L is the linearized Christoffel symbol

$$\Gamma^L_{\alpha\mu\nu} = \tfrac{1}{2}[\partial_\alpha\mathfrak{h}_{\mu\nu} - \partial_\mu\mathfrak{h}_{\nu\alpha} - \partial_\nu\mathfrak{h}_{\mu\alpha} + \tfrac{1}{2}(\eta_{\mu\alpha}\partial_\nu + \eta_{\nu\alpha}\partial_\mu - \eta_{\mu\nu}\partial_\alpha){\rm tr}[\mathfrak{h}\eta]]. \tag{61}$$

The full nonlinear action of general relativity is obtained by coupling the nonderivative part of $\tau_{\mathfrak{h}\mu\nu}$ as a source for $\mathfrak{h}^{\mu\nu}$,

$$S_{\rm GR} = S^L_{\rm GR} + \int d^4x\kappa\mathfrak{h}^{\mu\nu}(\Gamma^\beta{}_{\beta\alpha}\Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha}). \tag{62}$$

Variation of this action with respect to $\mathfrak{h}^{\mu\nu}$ yields the Einstein equation $R_{\mu\nu} = 0$ in the form

$$\kappa R^L_{\mu\nu}(\Gamma) = \tfrac{1}{2}\tau_{\mathfrak{h}\mu\nu} + \sigma_{\mu\nu}, \tag{63}$$

which implies

$$R^L_{\mu\nu}(\Gamma^L) = 8\pi G_N\tau_{\mathfrak{h}\mu\nu}. \tag{64}$$

This verifies that coupling the nonderivative part of $\tau_{\mathfrak{h}\mu\nu}$ as a source for $\mathfrak{h}^{\mu\nu}$ indeed produces the usual Einstein equations. Moreover, since the nonderivative part of $\tau_{\mathfrak{h}\mu\nu}$ is independent of $\eta^{\mu\nu}$, it generates no additional contribution to the energy-momentum tensor and so no further iteration steps are required.

B. Generic bootstrap results

In this subsection, we outline some generic applications of the bootstrap procedure, starting from an action given in Minkowski spacetime. The example relevant in our context is either an action $S^{(0)}$ independent of $\mathfrak{h}^{\mu\nu}$ or an action $S^{(1)}$ linear in $\mathfrak{h}^{\mu\nu}$. In each case, we seek to construct the corresponding action S that incorporates consistent self-coupling to $\mathfrak{h}^{\mu\nu}$ at all orders.

1. Case of $S^{(0)}$

Consider first the case of an action $S^{(0)}$ independent of $\mathfrak{h}^{\mu\nu}$, such as a matter action. We write

$$S^{(0)} = \int d^4x\mathcal{L}^{(0)}, \tag{65}$$

where the Lagrange density

$$\mathcal{L}^{(0)} = \mathcal{L}^{(0)}(\eta_{\mu\nu}, f_a, \partial_\mu f_a) \tag{66}$$

is a function of the spacetime metric $\eta_{\mu\nu}$, a set of fields $f_a(x)$, and their derivatives $\partial_\mu f_a$. For the purposes of this work, it suffices to suppose that the terms $\partial_\mu f_a$ are either derivatives of scalars or are gauge kinetic terms, so that promotion of ∂_μ to the auxiliary covariant derivative has no effect: $\partial_\mu f_a \to D_\mu[\psi]f_a \equiv \partial_\mu f_a$. This simplifying assumption avoids the need to consider terms of the $\sigma_{\mu\nu}$ type in the analysis.

To obtain the energy-momentum tensor for the action (65), the Lagrange density $\mathcal{L}^{(0)}$ is promoted to a covariant expression with respect to $\psi^{\mu\nu}$,

$$\mathcal{L}^{(0)} \to \mathcal{L}^{(0)}(\psi_{\mu\nu}, f_a, \partial_\mu f_a). \tag{67}$$

To ensure $\mathcal{L}^{(0)}$ remains a density, multiplication by a factor of a power of $\sqrt{|\psi|}$ may be required as part of this promotion, where $\psi \equiv \det[\psi^{\mu\nu}]$. Using the definition (56) then yields

$$-\frac{1}{2}\tau^{(0)}_{\mu\nu} = \frac{\delta\mathcal{L}^{(0)}}{\delta\psi^{\mu\nu}}\Big|_{\psi\to\eta}. \tag{68}$$

The bootstrap procedure requires that $\tau^{(0)}_{\mu\nu}$ be consistently coupled as a source for $\mathfrak{h}^{\mu\nu}$. The action $S^{(0)}$ must therefore be supplemented by an additional term

$$S^{(1)} = \int d^4x \mathcal{L}^{(1)} \equiv \int d^4x \mathfrak{h}^{\mu\nu}\Big(-\frac{1}{2}\tau^{(0)}_{\mu\nu}\Big), \tag{69}$$

up to a possible constant. However, in general the term $S^{(1)}$ itself contributes a term $\tau^{(1)}_{\mu\nu}$ to the energy-momentum tensor,

$$-\frac{1}{2}\tau^{(1)}_{\mu\nu} = \frac{\delta\mathcal{L}^{(1)}}{\delta\psi^{\mu\nu}}\Big|_{\psi\to\eta} = \mathfrak{h}^{\alpha\beta}\frac{\delta(-\frac{1}{2}\tau^{(0)}_{\alpha\beta})}{\delta\psi^{\mu\nu}}\Big|_{\psi\to\eta}. \tag{70}$$

Consistency of the coupling then requires that a further term $S^{(2)}$ be added to the action,

$$S^{(2)} = \int d^4x \mathcal{L}^{(2)}, \tag{71}$$

where $\mathcal{L}^{(2)}$ is the solution to the differential equation

$$\frac{\delta\mathcal{L}^{(2)}}{\delta\mathfrak{h}^{\mu\nu}}\Big|_{\psi\to\eta} = -\frac{1}{2}\tau^{(1)}_{\mu\nu} \equiv \mathfrak{h}^{\alpha\beta}\frac{\delta(-\frac{1}{2}\tau^{(0)}_{\alpha\beta})}{\delta\psi^{\mu\nu}}\Big|_{\psi\to\eta}. \tag{72}$$

We find

$$\mathcal{L}^{(2)} = \frac{1}{2}\mathfrak{h}^{\alpha\beta}\mathfrak{h}^{\gamma\delta}\frac{\delta(-\frac{1}{2}\tau^{(0)}_{\gamma\delta})}{\delta\psi^{\alpha\beta}}\Big|_{\psi\to\eta} = \frac{1}{2}\mathfrak{h}^{\mu\nu}\Big(-\frac{1}{2}\tau^{(1)}_{\mu\nu}\Big), \tag{73}$$

up to a possible constant.

Iterating this procedure yields a series of terms summing to the desired Lagrange density $\mathcal{L}$,

$$\mathcal{L} = \sum_{n=0}^{\infty}\frac{1}{n!}\mathfrak{h}^{\alpha_1\beta_1}\cdots\mathfrak{h}^{\alpha_n\beta_n}\frac{\delta^n(-\frac{1}{2}\tau^{(0)}_{\alpha_n\beta_n})}{\delta\psi^{\alpha_1\beta_1}\cdots\delta\psi^{\alpha_{n-1}\beta_{n-1}}}\Big|_{\psi\to\eta}. \tag{74}$$

The series can be constructed provided the integrability conditions are satisfied at each step, and it may terminate at some finite n. It represents a Taylor expansion of $\mathcal{L}$, and inspection reveals the identification

$$\mathcal{L} = \mathcal{L}^{(0)}(\psi_{\mu\nu}, f_a, \partial_\mu f_a)|_{\psi\to\mathfrak{g}}. \tag{75}$$

The above derivation shows that knowledge of $\mathcal{L}^{(0)}$ in the form (66) suffices to determine $\mathcal{L}$. If originally the matter-gravity coupling is specified in the linearized form (69), the bootstrap procedure amounts to finding $\mathcal{L}^{(0)}$ and then determining $\mathcal{L}$ via Eqs. (67) and (75). If instead a pure matter action is specified by giving $\mathcal{L}^{(0)}$, it suffices to promote it according to Eq. (67) and obtain $\mathcal{L}$ via the identification (75). In this case, the bootstrap corresponds to the standard minimal-coupling procedure. For example, the usual Minkowski-spacetime energy-momentum tensor for Maxwell electrodynamics is

$$T^{\rm EM}_{\mu\nu} = F_\mu{}^\lambda F_{\nu\lambda} - \tfrac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} = \tau^{(0)}_{{\rm EM}\mu\nu}, \tag{76}$$

with the latter equality following from conformal invariance. The corresponding Lagrange density is

$$\mathcal{L}^{(0)}_{\rm EM} = -\tfrac{1}{4}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}. \tag{77}$$

Promoting this according to Eq. (67) and making the identification (75) directly yields the usual Lagrange density $\mathcal{L}_{\rm EM}$ for electrodynamics in curved spacetime,

$$\mathcal{L}_{\rm EM} = -\frac{1}{4\sqrt{|\mathfrak{g}|}}\mathfrak{g}^{\alpha\gamma}\mathfrak{g}^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}, \tag{78}$$

where $\mathfrak{g} \equiv \det[\mathfrak{g}^{\mu\nu}]$.

2. Case of $S^{(1)}$

Under some circumstances, the given starting point is instead an action $S^{(1)}$ for a theory linear in $\mathfrak{h}^{\mu\nu}$. To obtain the fully coupled action S, one can explicitly perform the iteration procedure above. However, a more efficient "inverse" method can be adopted instead. To implement this method, we start by identifying the energy-momentum tensor $\tau^{(0)}_{\mu\nu}$ from the specified action $S^{(1)}$ written in the form (69), and we promote it to a covariant expression with respect to $\psi^{\mu\nu}$:

$$\tau^{(0)}_{\mu\nu}(\eta) \to \tau^{(0)}_{\mu\nu}(\psi). \tag{79}$$

An appropriate multiplicative factor of $\sqrt{|\psi|}$ may be required to maintain the tensor transformation properties of $\tau^{(0)}_{\mu\nu}$. We then write the differential equation

$$-\frac{1}{2}\tau^{(0)}_{\mu\nu} = \frac{\delta\mathcal{L}^{(0)}}{\delta\psi^{\mu\nu}}, \tag{80}$$

which reduces to Eq. (68) in the limit $\psi^{\mu\nu} \to \eta^{\mu\nu}$. The differential equation can be solved if the integrability condition

$$\frac{\delta\tau^{(0)}_{\mu\nu}}{\delta\psi^{\alpha\beta}} = \frac{\delta\tau^{(0)}_{\alpha\beta}}{\delta\psi^{\mu\nu}} \tag{81}$$

is satisfied. Once the solution $\mathcal{L}^{(0)}$ is obtained, we can apply the identification (75) to obtain $\mathcal{L}$ and hence S.

The above inverse trick is applied in some of the analysis that follows. To illustrate it in a more familiar context, consider the cosmological constant Λ. In Minkowski spacetime, Λ is associated with an effective energy-momentum tensor given by

$$T^{\Lambda}_{\mu\nu} = -2\kappa\Lambda\eta_{\mu\nu} = -\tau^{(0)}_{\Lambda\mu\nu}. \tag{82}$$

The challenge is to bootstrap this to the fully coupled Lagrange density $\mathcal{L}^{\Lambda}$. Following the inverse trick, we promote $\tau^{(0)}_{\Lambda\mu\nu}$ to

$$\tau^{(0)}_{\Lambda\mu\nu}(\psi) = 2\kappa\Lambda\sqrt{|\psi|}\psi_{\mu\nu}, \tag{83}$$

where the appropriate factor of $\sqrt{|\psi|}$ has been introduced. With the identities

$$\begin{aligned} \delta\psi_{\mu\nu} &= -\psi_{\mu\alpha}\psi_{\nu\beta}\delta\psi^{\alpha\beta}, \\ \delta\sqrt{|\psi|} &= \tfrac{1}{2}\sqrt{|\psi|}\psi_{\alpha\beta}\delta\psi^{\alpha\beta}, \end{aligned} \tag{84}$$

the integrability condition (81) can be verified, so the differential equation (80) can be solved for $\mathcal{L}^{(0)}(\psi)$. Making the identification (75) then yields

$$\mathcal{L}^{\Lambda} = -2\kappa\Lambda\sqrt{|\mathfrak{g}|}, \tag{85}$$

in agreement with the usual result. Notice that the linearized version of this is

$$\mathcal{L}^{\Lambda} \approx -2\kappa\Lambda - \kappa\Lambda\mathfrak{h}^{\mu\nu}\eta_{\mu\nu} = -2\kappa\Lambda + \mathfrak{h}^{\mu\nu}(-\tfrac{1}{2}\tau^{(0)}_{\Lambda\mu\nu}), \tag{86}$$

as expected from Eq. (82), and that the zeroth-order term $\mathcal{L}^{(0)}(\eta)$ is merely a constant in this example. Note also that the first-order term $\mathcal{L}^{(1)}(\eta)$ produces a linear instability in the action at this order. This could be avoided by initiating the bootstrap from a theory formulated in a suitable Riemann background spacetime [17].

As another example, consider the bootstrap procedure for the transverse-traceless (TT) gauge. A common form for this gauge involves the trace-corrected field $\bar{h}^{\mu\nu}$ and a timelike unit vector n_μ:

$$\mathrm{tr}[\bar{h}\eta] = 0, \qquad n_\mu\bar{h}^{\mu\nu} = 0, \qquad \partial_\mu\bar{h}^{\mu\nu} = 0. \tag{87}$$

These standard linear gauge-fixing conditions can be expressed in terms of $\mathfrak{h}^{\mu\nu}$ and $\Gamma^{L\alpha}{}_{\mu\nu}$ using Eqs. (53) and (61). The resulting expressions can then be implemented in the linearized action (54) via the addition of the linear Lagrange density

$$\mathcal{L}^{L}_{\mathrm{TT}} = \lambda_{(1)}\,\mathrm{tr}[\mathfrak{h}\eta] + \lambda_{(2)\nu}n_\mu\mathfrak{h}^{\mu\nu} + \lambda_{(3)\alpha}\eta^{\mu\nu}\Gamma^{L\alpha}{}_{\mu\nu}, \tag{88}$$

where $\lambda_{(1)}$, $\lambda_{(2)\nu}$, and $\lambda_{(3)\alpha}$ are Lagrange multipliers. The bootstrap procedure can be applied to each of the three terms independently. The first term is linear in $\mathfrak{h}^{\mu\nu}$ and of the same form as in Eq. (86), so the bootstrap is immediate. The second term is also linear in $\mathfrak{h}^{\mu\nu}$, and the integrability conditions are directly satisfied. The inverse trick described above can therefore be applied. The third term is independent of $\mathfrak{h}^{\mu\nu}$, so the bootstrap method of the previous subsection applies. The net result of the bootstrap is the nonlinear constraint terms

$$\begin{aligned} \mathcal{L}_{\mathrm{TT}} = {}& 2\lambda_{(1)}(\sqrt{|\mathfrak{g}|} - \sqrt{|\eta|}) + \lambda_{(2)\nu}n_\mu(\mathfrak{g}^{\mu\nu} - \eta^{\mu\nu}) \\ & + \lambda_{(3)\alpha}\mathfrak{g}^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu}, \end{aligned} \tag{89}$$

which correspond to a nonlinear form of the TT gauge constraints,

$$\sqrt{|\mathfrak{g}|} = \sqrt{|\eta|}, \qquad n_\mu\mathfrak{g}^{\mu\nu} = n_\mu\eta^{\mu\nu}, \qquad \mathfrak{g}^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = 0. \tag{90}$$

IV. BOOTSTRAP FOR CARDINAL GRAVITY

At this stage, we are in a position to consider the nonlinear extension of the cardinal theory (1). This section begins by presenting a convenient first-order reformulation of the linear cardinal theory. In this form, the bootstrap of the kinetic terms is straightforward using the methods of the previous section. We investigate the bootstrap integrability conditions on an arbitrary potential term, which turn out to provide interesting constraints on the theory. Finally, the bootstrap of these terms is also presented.

A. First-order action

To facilitate comparison with the bootstrap for general relativity, a first-order form of the theory (1) is useful. To develop this, we introduce the trace-reversed cardinal field $\mathfrak{C}^{\mu\nu}$ as

$$\mathfrak{C}^{\mu\nu} = -C^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}C^{\alpha}{}_{\alpha}. \tag{91}$$

Note the signs, which are chosen to improve the correspondence to the conventions used in the analysis of general relativity. The field $\mathfrak{C}^{\mu\nu}$ plays a central role in what follows.

In terms of $\mathfrak{C}^{\mu\nu}$, the second-order Lagrange density $\mathcal{L}_{\mathfrak{C}}$ yielding equivalent equations of motion to the theory (1) takes the form

$$\mathcal{L}_{\mathfrak{C}} = \tfrac{1}{2}\mathfrak{C}^{\mu\nu}\mathfrak{K}_{\mu\nu\alpha\beta}\mathfrak{C}^{\alpha\beta} - \mathfrak{V}(\mathfrak{C}^{\mu\nu}, \eta_{\mu\nu}). \tag{92}$$

Here, the quadratic operator $\mathfrak{K}_{\mu\nu\alpha\beta}$ is given in Cartesian coordinates by

$$\begin{aligned} \mathfrak{K}_{\mu\nu\alpha\beta} = {}& \tfrac{1}{4}\kappa[-(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})\partial^\lambda\partial_\lambda \\ & - \eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\alpha\beta}\partial_\mu\partial_\nu \\ & + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\alpha}\partial_\mu\partial_\beta \\ & + \eta_{\mu\beta}\partial_\nu\partial_\alpha + \eta_{\nu\beta}\partial_\mu\partial_\alpha]. \end{aligned} \tag{93}$$

Note that acting with this operator on the fluctuation $\mathfrak{h}^{\alpha\beta}$ produces the linearized Ricci curvature $R^{L}_{\mu\nu}$:

$$\mathfrak{K}_{\mu\nu\alpha\beta}\mathfrak{h}^{\alpha\beta} \equiv \kappa R^{L}_{\mu\nu}. \tag{94}$$

Note also that the quantities $K_{\mu\nu\alpha\beta}C^{\alpha\beta}$ in Eq. (1) and $\mathfrak{K}_{\mu\nu\alpha\beta}\mathfrak{C}^{\alpha\beta}$ are related by trace reversal with a sign. In Eq. (92), the potential $\mathfrak{V}(\mathfrak{C}^{\mu\nu}, \eta_{\mu\nu})$ is determined by the requirement that the equations of motion

$$\mathfrak{K}_{\mu\nu\alpha\beta}\mathfrak{C}^{\alpha\beta} - \frac{\delta\mathfrak{V}}{\delta\mathfrak{C}^{\mu\nu}} = 0 \tag{95}$$

have the same content as the original equations of motion (3). This requires that

$$\frac{\delta\mathfrak{V}}{\delta\mathfrak{C}^{\mu\nu}} = -\frac{\delta V}{\delta C^{\mu\nu}} + \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\frac{\delta V}{\delta C^{\alpha\beta}}. \tag{96}$$

To construct the first-order form of the linear cardinal theory, we follow a similar path to that of the Palatini formalism in general relativity discussed in Sec. III A. Introducing an independent auxiliary field $\Gamma^{\alpha}{}_{\mu\nu}$, the Lagrange density (92) can be rewritten in terms of $\mathfrak{C}^{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu}$ in the equivalent form

$$\begin{aligned} S^L_{\mathfrak{C}} &= \int d^4x \mathcal{L}^L_{\mathfrak{C}}, \\ \mathcal{L}^L_{\mathfrak{C}} &= \kappa[\mathfrak{C}^{\mu\nu}(\partial_\alpha\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}{}_{\mu\alpha}) \\ &\quad + \eta^{\mu\nu}(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha})] + \mathfrak{V} \\ &\equiv \mathfrak{K}^L + \mathfrak{V}, \end{aligned} \tag{97}$$

where $\mathfrak{K}^L$ is the kinetic part of the Lagrange density. Variation of this action with respect to the independent fields $\mathfrak{C}^{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu}$ gives the equations of motion. With standard manipulations, the equations of motion determine the fields $\Gamma^{\alpha}{}_{\mu\nu}$ to be linearized Christoffel symbols of the conventional form but depending on $\mathfrak{C}^{\mu\nu}$ instead of $\mathfrak{h}^{\mu\nu}$. They also imply linearized versions of the equations of motion (95) for $\mathfrak{C}^{\mu\nu}$ obtained from the second-order Lagrange density (92).

The linearized action (97) can be written in other equivalent forms by decomposing the cardinal field $\mathfrak{C}^{\mu\nu}$. In the minimum of the potential $\mathfrak{V}$, the field $\mathfrak{C}^{\mu\nu}$ acquires an expectation value $\mathfrak{c}^{\mu\nu}$,

$$\langle\mathfrak{C}^{\mu\nu}\rangle = \mathfrak{c}^{\mu\nu} \equiv -c^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}c^{\alpha}{}_{\alpha}. \tag{98}$$

This satisfies the identities

$$\begin{aligned} \mathrm{tr}[\mathfrak{c}\eta] &= \mathrm{tr}[c\eta], \\ \mathrm{tr}[(\mathfrak{c}\eta)^2] &= \mathrm{tr}[(c\eta)^2], \\ \mathrm{tr}[(\mathfrak{c}\eta)^3] &= -\mathrm{tr}[(c\eta)^3] + \tfrac{3}{2}\mathrm{tr}[c\eta]\,\mathrm{tr}[(c\eta)^2] - \tfrac{1}{4}(\mathrm{tr}[c\eta])^3, \\ \mathrm{tr}[(\mathfrak{c}\eta)^4] &= \mathrm{tr}[(c\eta)^4] - 2\mathrm{tr}[c\eta]\,\mathrm{tr}[(c\eta)^3] \\ &\quad + \tfrac{3}{2}(\mathrm{tr}[c\eta])^2\,\mathrm{tr}[(c\eta)^2] - \tfrac{1}{4}(\mathrm{tr}[c\eta])^4, \end{aligned} \tag{99}$$

and it also obeys

$$\partial_\alpha\mathfrak{c}^{\mu\nu} = 0 \tag{100}$$

by virtue of the assumption (10). The fluctuation $\tilde{\mathfrak{C}}^{\mu\nu}$ about $\mathfrak{c}^{\mu\nu}$ is

$$\tilde{\mathfrak{C}}^{\mu\nu} = -\tilde{C}^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}\tilde{C}^{\alpha}{}_{\alpha}. \tag{101}$$

The analogue of Eq. (11) therefore becomes

$$\mathfrak{C}^{\mu\nu} = \mathfrak{c}^{\mu\nu} + \tilde{\mathfrak{C}}^{\mu\nu}. \tag{102}$$

An alternative expression for the linearized action (97) is therefore

$$\begin{aligned} S^L_{\tilde{\mathfrak{C}}} &= \int d^4x \mathcal{L}^L_{\tilde{\mathfrak{C}}}, \\ \mathcal{L}^L_{\tilde{\mathfrak{C}}} &= \kappa[\tilde{\mathfrak{C}}^{\mu\nu}(\partial_\alpha\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}{}_{\mu\alpha}) \\ &\quad + \eta^{\mu\nu}(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha})] + \mathfrak{V} \\ &\equiv \mathfrak{K}^L_{\tilde{\mathfrak{C}}} + \mathfrak{V}. \end{aligned} \tag{103}$$

Note that the two linearized actions $S^L_{\mathfrak{C}}$ and $S^L_{\tilde{\mathfrak{C}}}$ are identical, but by virtue of Eq. (100) the kinetic term $\mathfrak{K}^L$ differs from $\mathfrak{K}^L_{\tilde{\mathfrak{C}}}$ by a total derivative.

The cardinal field $\mathfrak{C}^{\mu\nu}$ can be further decomposed into NG modes and massive modes, in parallel with Eq. (23). We write

$$\mathfrak{C}^{\mu\nu} = \mathfrak{c}^{\mu\nu} + \mathfrak{N}^{\mu\nu} + \mathfrak{M}^{\mu\nu}, \tag{104}$$

where the trace-reversed NG field $\mathfrak{N}^{\mu\nu}$ is defined as

$$\mathfrak{N}^{\mu\nu} = -N^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}N^{\alpha}{}_{\alpha} \tag{105}$$

and the trace-reversed massive-mode field is

$$\mathfrak{M}^{\mu\nu} = -M^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}M^{\alpha}{}_{\alpha}. \tag{106}$$

The constraints in the NG sector corresponding to Eq. (15) can be written as

$$\mathrm{tr}[\mathfrak{N}\eta(c\eta)^j] = 0, \tag{107}$$

with $j = 0, 1, 2, 3$, while the analogue of Eq. (19) is

$$\mathrm{tr}[\mathfrak{N}\eta F(\mathfrak{c}\eta, \mathfrak{M}\eta)] = 0, \tag{108}$$

where $F(\mathfrak{c}\eta, \mathfrak{M}\eta)$ is an arbitrary matrix polynomial in $\mathfrak{c}\eta$ and $\mathfrak{M}\eta$. Another equivalent form for the action (97) is therefore

$$\begin{aligned} S^L_{\mathfrak{N},\mathfrak{M}} &= \int d^4x \mathcal{L}^L_{\mathfrak{N},\mathfrak{M}}, \\ \mathcal{L}^L_{\mathfrak{N},\mathfrak{M}} &= \kappa[(\mathfrak{N}^{\mu\nu} + \mathfrak{M}^{\mu\nu})(\partial_\alpha\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}{}_{\mu\alpha}) \\ &\quad + \eta^{\mu\nu}(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha})] + \mathfrak{V} \\ &= K^L_{\mathfrak{N},\mathfrak{M}} + \mathfrak{V}, \end{aligned} \tag{109}$$

where $\mathfrak{K}^L_{\mathfrak{N},\mathfrak{M}}$ denotes the kinetic term expressed in terms of $\mathfrak{N}^{\mu\nu}$, $\mathfrak{M}^{\mu\nu}$, and $\Gamma^{\alpha}{}_{\mu\nu}$.

B. Kinetic bootstrap

With the linear cardinal theory massaged into a first-order form paralleling that used for general relativity, we are in a position to investigate the bootstrap to nonlinear cardinal gravity. Since the bootstrap involves adding self-coupling order by order, it can be done independently for each part in the action. In particular, the bootstrap for the kinetic part parallels the bootstrap for the linearized version (54) of general relativity.

1. Primary bootstrap

It is perhaps most natural to apply the bootstrap procedure to the linearized theory in the form (97), which holds prior to the spontaneous Lorentz breaking. For the corresponding kinetic term $\mathfrak{K}^L$, the energy-momentum tensor associated with $\mathfrak{C}^{\mu\nu}$ is of the same form as before,

$$-\tfrac{1}{2}(\tau_{\mathfrak{C}})_{\mu\nu} = \kappa(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha}) + \kappa\sigma_{\mu\nu}, \tag{110}$$

and the nonlinear kinetic action $S_{\mathfrak{K},\mathfrak{C}}$ is obtained by coupling its nonderivative part as a source for $\mathfrak{C}^{\mu\nu}$,

$$\begin{aligned} S_{\mathfrak{K},\mathfrak{C}} &= S^L_{\mathfrak{K},\mathfrak{C}} + \int d^4x\kappa\mathfrak{C}^{\mu\nu}(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha}) \\ &= \int d^4x\kappa(\eta^{\mu\nu} + \mathfrak{C}^{\mu\nu})R_{\mu\nu}(\Gamma), \end{aligned} \tag{111}$$

where $R_{\mu\nu}(\Gamma)$ is the Ricci curvature defined via the auxiliary field $\Gamma^{\alpha}{}_{\mu\nu}$ in the usual way,

$$\begin{aligned} R_{\mu\nu}(\Gamma) = \partial_\alpha\Gamma^{\alpha}{}_{\mu\nu} - \tfrac{1}{2}\partial_\mu\Gamma^{\alpha}{}_{\nu\alpha} - \tfrac{1}{2}\partial_\nu\Gamma^{\alpha}{}_{\mu\alpha} \\ + (\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha}). \end{aligned} \tag{112}$$

Since the extra term in Eq. (111) is independent of $\eta^{\mu\nu}$, no further iteration steps are needed.

In the extremum of the potential $\mathfrak{V}$, the massive modes vanish and the result (111) for the kinetic bootstrap reduces to

$$S_{\mathfrak{K},\mathfrak{C}} \supset \int d^4x\kappa(\eta^{\mu\nu} + \mathfrak{c}^{\mu\nu} + \mathfrak{N}^{\mu\nu})R_{\mu\nu}(\Gamma). \tag{113}$$

The combination $(\eta^{\mu\nu} + \mathfrak{c}^{\mu\nu})$ can be viewed as playing the role of an effective background metric. Under a suitable change of coordinates, this effective metric can be brought to the Minkowski form, $(\eta^{\mu\nu} + \mathfrak{c}^{\mu\nu}) \to \eta^{\mu\nu}$. With the identification

$$\mathfrak{h}^{\mu\nu} \leftrightarrow \mathfrak{N}^{\mu\nu}, \tag{114}$$

which matches the linearized correspondence (47), it follows that the kinetic action $S_{\mathfrak{K},\mathfrak{C}}$ reduces to the Einstein-Hilbert action in the limit of vanishing massive modes. The result (111) for the kinetic bootstrap thereby reveals that the nonlinear cardinal theory represents an alternative theory of gravity containing general relativity in a suitable low-energy limit. The correspondence

$$\mathfrak{g}^{\mu\nu} \leftrightarrow \eta^{\mu\nu} + \tilde{\mathfrak{C}}^{\mu\nu} \tag{115}$$

provides the match between the metric density $\mathfrak{g}^{\mu\nu}$ of general relativity and fields in cardinal gravity.

2. Alternative bootstraps

The derivation of the action $S_{\mathfrak{K},\mathfrak{C}}$ in Eq. (111) is based on applying the bootstrap to the linearized cardinal action (97) for the cardinal field $\mathfrak{C}^{\mu\nu}$. However, the spontaneous Lorentz violation produces a phase transition that naturally separates the cardinal excitations into NG and massive modes. One could therefore instead consider applying the bootstrap to various choices of excitation in the effective theory describing the physics after the spontaneous symmetry breaking has occurred. In the remainder of this subsection, we consider these alternative bootstrap procedures and their application to the kinetic term in the linear cardinal theory.

Suppose the bootstrap is instead applied to the alternative linearized cardinal action (103) for the fluctuation $\tilde{\mathfrak{C}}^{\mu\nu}$. This procedure has the possible disadvantage of requiring a preestablished value for the vacuum expectation $\mathfrak{c}^{\mu\nu}$. However, since $\tilde{\mathfrak{C}}^{\mu\nu}$ is a fluctuation, this procedure does parallel more closely the usual bootstrap in general relativity, for which the relevant field $\mathfrak{h}^{\mu\nu}$ is also a fluctuation. The derivation of the nonlinear action $S_{\mathfrak{K},\tilde{\mathfrak{C}}}$ from the linearized theory (103) proceeds as before. The result for this secondary theory is

$$\begin{aligned} S_{\mathfrak{K},\tilde{\mathfrak{C}}} &= S^L_{\mathfrak{K},\tilde{\mathfrak{C}}} + \int d^4x\kappa\tilde{\mathfrak{C}}^{\mu\nu}(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha}) \\ &= \int d^4x\kappa(\eta^{\mu\nu} + \tilde{\mathfrak{C}}^{\mu\nu})R_{\mu\nu}(\Gamma). \end{aligned} \tag{116}$$

This is equivalent to the action $S_{\mathfrak{K},\mathfrak{C}}$ under a suitable coordinate transformation. We thereby find that the secondary bootstrap yields the same physics for the kinetic term as did the primary bootstrap leading to Eq. (111).

A tertiary theory could also be countenanced, in which the bootstrap is applied only to the NG modes $\mathfrak{N}^{\mu\nu}$ appearing in the linearized action (109). While this procedure also requires a preestablished value for the vacuum expectation $\mathfrak{c}^{\mu\nu}$, it has the possible advantage of matching more closely the symmetry structure of the bootstrap for general relativity. The key point is that the gauge transformation (5), which fails to be a symmetry of the linearized theory due to the potential, nonetheless does define a symmetry for the pure NG sector because the potential vanishes for pure NG excitations. In linearized general relativity, the analogous gauge symmetry can be related to the conserved two-tensor current, and it morphs into diffeomorphism symmetry following the bootstrap procedure. In the present context, this symmetry structure is reproduced in the pure NG sector if the bootstrap is applied only to the NG excitation $\mathfrak{N}^{\mu\nu}$ in the linearized action (109).

For this tertiary bootstrap, the first step is to obtain the energy-momentum tensor for the kinetic term $K^L_{\mathfrak{N},\mathfrak{M}}$ in terms of the NG and massive modes. The calculations for this step again parallel those for the bootstrap in general relativity. We find

$$-\tfrac{1}{2}(\tau_{\mathfrak{N},\mathfrak{M}})_{\mu\nu} = \kappa(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha}) + \kappa\sigma_{\mu\nu}(\mathfrak{N},\Gamma) + \kappa\sigma_{\mu\nu}(\mathfrak{M},\Gamma), \quad (117)$$

where $\sigma_{\mu\nu}$ is the total-derivative term given by Eq. (58) but with modified arguments as indicated. The prescription for the tertiary bootstrap is then to couple the nonderivative part of $(\tau_{\mathfrak{N},\mathfrak{M}})_{\mu\nu}$ as a source for $\mathfrak{N}^{\mu\nu}$,

$$\mathfrak{K}_{\mathfrak{N},\mathfrak{M}} = \mathfrak{K}^{L}_{\mathfrak{N},\mathfrak{M}} + \kappa\mathfrak{N}^{\mu\nu}(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha}). \quad (118)$$

This prescription yields the tertiary kinetic action

$$S_{\mathfrak{K},\mathfrak{N},\mathfrak{M}} = \int d^4x\kappa(\eta^{\mu\nu} + \mathfrak{N}^{\mu\nu})R_{\mu\nu}(\Gamma) + \kappa\mathfrak{M}^{\mu\nu}(\partial_{\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\alpha}). \quad (119)$$

Paralleling the case of general relativity, the extra term in Eq. (118) is independent of $\eta^{\mu\nu}$, so no further iteration steps are needed. Note that the structure of this result implies the auxiliary field $\Gamma^{\alpha}{}_{\mu\nu}$ is no longer equivalent on shell to a Christoffel symbol.

The tertiary kinetic action $S_{\mathfrak{K},\mathfrak{N},\mathfrak{M}}$ differs nontrivially from the primary one $S_{\mathfrak{K},\mathfrak{C}}$, and the physical content of the two is also different. With the identification (114) and in the pure NG sector, both actions match the Einstein-Hilbert action of general relativity. Their linearized content is also the same as that of the linear cardinal theory (1).

C. Integrability conditions for the potential

Next, we investigate the integrability conditions required to apply the bootstrap on the potential term. We obtain constraints such that $\mathfrak{V}$ obeys the integrability conditions, and we determine a general form of $\mathfrak{V}$ satisfying these constraints.

To proceed, start with the theory in the form (92) in terms of the cardinal field $\mathfrak{C}^{\mu\nu}$. The potential is $\mathfrak{V}(\mathfrak{C}^{\mu\nu}, \eta_{\mu\nu})$, and it is a scalar. The only scalars that can be formed from $\mathfrak{C}^{\mu\nu}$ and $\eta_{\mu\nu}$ involve traces of the matrix $\mathfrak{C}\eta$. The scalar $\mathfrak{X}_m$ with m such products has the form

$$\mathfrak{X}_m = \mathrm{tr}[(\mathfrak{C}\eta)^m]. \quad (120)$$

Since $\mathfrak{C}\eta$ is a 4×4 matrix, only four of these are independent, so the potential $\mathfrak{V}(\mathfrak{C}^{\mu\nu}, \eta_{\mu\nu})$ can be written

$$\mathfrak{V}(\mathfrak{C}^{\mu\nu}, \eta_{\mu\nu}) = \mathfrak{V}(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4). \quad (121)$$

In the minimum of $\mathfrak{V}$, $\mathfrak{C}^{\mu\nu} = \mathfrak{c}^{\mu\nu}$ and the scalars $\mathfrak{X}_m$ have expectation values

$$\langle\mathfrak{X}_m\rangle = \mathrm{tr}[(\mathfrak{c}\eta)^m] \equiv \mathfrak{x}_m. \quad (122)$$

The next step is to determine the energy-momentum tensor $\tau_{\mathfrak{C}\mu\nu}$ associated with the potential $\mathfrak{V}$ and check the integrability conditions. We therefore promote $\mathfrak{V}$ to a covariant expression with respect to the auxiliary metric density $\psi^{\alpha\beta}$,

$$\mathfrak{V}(\mathfrak{C}^{\mu\nu}, \eta_{\mu\nu}) \to \sqrt{|\psi|}\mathfrak{V}(\mathfrak{C}^{\mu\nu}/\sqrt{|\psi|}, \sqrt{|\psi|}\psi_{\mu\nu}) = \sqrt{|\psi|}\mathfrak{V}(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4), \quad (123)$$

where the four quantities $\mathfrak{X}_m$ are now

$$\mathfrak{X}_m(\psi) = \mathrm{tr}[(\mathfrak{C}\psi)^m] \quad (124)$$

and are scalars with respect to $\psi^{\mu\nu}$. In parallel with the bootstrap for the kinetic term, $\mathfrak{C}^{\mu\nu}$ is taken to be a tensor density with respect to $\psi^{\mu\nu}$ in constructing these expressions.

The energy-momentum tensor $\tau_{\mathfrak{C}\mu\nu}$ is

$$-\frac{1}{2}\tau_{\mathfrak{C}\mu\nu} = \frac{\delta(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\mu\nu}}. \quad (125)$$

The bootstrap procedure requires this to be obtained from an action by varying with respect to $\mathfrak{C}^{\mu\nu}$. We must therefore add to the Lagrange density a term $\mathfrak{V}'$ such that

$$\frac{\delta\mathfrak{V}'}{\delta\mathfrak{C}^{\mu\nu}} = -\frac{1}{2}\tau_{C\mu\nu} = \frac{\delta(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\mu\nu}}. \quad (126)$$

If $\mathfrak{V}'$ is smooth, then

$$\frac{\delta^2\mathfrak{V}'}{\delta\mathfrak{C}^{\mu\nu}\delta\mathfrak{C}^{\alpha\beta}} = \frac{\delta^2\mathfrak{V}'}{\delta\mathfrak{C}^{\alpha\beta}\delta\mathfrak{C}^{\mu\nu}}, \quad (127)$$

which implies

$$\frac{\delta^2(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\mu\nu}\delta\mathfrak{C}^{\alpha\beta}} = \frac{\delta^2(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\alpha\beta}\delta\mathfrak{C}^{\mu\nu}}. \quad (128)$$

This is the integrability condition for the existence of $\mathfrak{V}'$. It requires symmetry of the double partial derivative under the interchange $(\mu\nu) \leftrightarrow (\alpha\beta)$.

The double derivative appearing in the result (128) can be written as

$$\frac{\delta^2(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\mu\nu}\delta\mathfrak{C}^{\alpha\beta}} = \sqrt{|\psi|}(A_{m\mu\nu\alpha\beta}\mathfrak{V}_m + B_{mn\mu\nu\alpha\beta}\mathfrak{V}_{mn}), \quad (129)$$

where m and n are summed, with

$$\mathfrak{V}_m \equiv \frac{\delta\mathfrak{V}}{\delta\mathfrak{X}_m}, \qquad \mathfrak{V}_{mn} \equiv \frac{\delta^2\mathfrak{V}}{\delta\mathfrak{X}_m\delta\mathfrak{X}_n}, \quad (130)$$

and with the coefficients $A_{m\mu\nu\alpha\beta}$ and $B_{mn\mu\nu\alpha\beta}$ given by

$$\begin{aligned}
A_{m\mu\nu\alpha\beta} &= \frac{1}{2}\psi_{\mu\nu}\frac{\delta\mathfrak{X}_m}{\delta\mathfrak{C}^{\alpha\beta}} + \frac{\delta^2\mathfrak{X}_m}{\delta\psi^{\mu\nu}\delta\mathfrak{C}^{\alpha\beta}} \\
&= \frac{1}{2}m\psi_{\mu\nu}[\psi(\mathfrak{C}\psi)^{m-1}]_{\alpha\beta} \\
&\quad - m\sum_{k=0}^{m-1}[\psi(\mathfrak{C}\psi)^{k}]_{\mu\alpha}[\psi(\mathfrak{C}\psi)^{m-1-k}]_{\nu\beta}, \\
B_{mn\mu\nu\alpha\beta} &= \frac{1}{2}\left(\frac{\delta\mathfrak{X}_m}{\delta\psi^{\mu\nu}}\frac{\delta\mathfrak{X}_n}{\delta\mathfrak{C}^{\alpha\beta}} + \frac{\delta\mathfrak{X}_n}{\delta\psi^{\mu\nu}}\frac{\delta\mathfrak{X}_m}{\delta\mathfrak{C}^{\alpha\beta}}\right) \\
&= -\frac{1}{2}mn([\psi(\mathfrak{C}\psi)^{m}]_{\mu\nu}[\psi(\mathfrak{C}\psi)^{n-1}]_{\alpha\beta} \\
&\quad + [\psi(\mathfrak{C}\psi)^{n}]_{\mu\nu}[\psi(\mathfrak{C}\psi)^{m-1}]_{\alpha\beta}). \qquad (131)
\end{aligned}$$

Inspection of these results reveals that the integrability condition is satisfied if and only if the combined quantity

$$\begin{aligned}
C_{mn\mu\nu\alpha\beta} &= \tfrac{1}{2}m\mathfrak{V}_m\psi_{\mu\nu}[\psi(\mathfrak{C}\psi)^{m-1}]_{\alpha\beta} \\
&\quad - mn\mathfrak{V}_{mn}[\psi(\mathfrak{C}\psi)^{m}]_{\mu\nu}[\psi(\mathfrak{C}\psi)^{n-1}]_{\alpha\beta} \qquad (132)
\end{aligned}$$

is symmetric under the interchange $(\mu\nu) \leftrightarrow (\alpha\beta)$.

Using the Hamilton-Cayley theorem, we can write

$$\begin{aligned}
[\psi(\mathfrak{C}\psi)^4]_{\mu\nu} &= p_1[\psi(\mathfrak{C}\psi)^3]_{\mu\nu} - p_2[\psi(\mathfrak{C}\psi)^2]_{\mu\nu} \\
&\quad + p_3[\psi\mathfrak{C}\psi]_{\mu\nu} - p_4\psi_{\mu\nu}, \qquad (133)
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= \mathfrak{X}_1, \\
p_2 &= \tfrac{1}{2}\mathfrak{X}_1^2 - \tfrac{1}{2}\mathfrak{X}_2, \\
p_3 &= \tfrac{1}{6}\mathfrak{X}_1^3 - \tfrac{1}{2}\mathfrak{X}_1\mathfrak{X}_2 + \tfrac{1}{3}\mathfrak{X}_3, \\
p_4 &= \tfrac{1}{24}\mathfrak{X}_1^4 - \tfrac{1}{4}\mathfrak{X}_1^2\mathfrak{X}_2 + \tfrac{1}{8}\mathfrak{X}_2^2 + \tfrac{1}{3}\mathfrak{X}_1\mathfrak{X}_3 - \tfrac{1}{4}\mathfrak{X}_4. \qquad (134)
\end{aligned}$$

Adopting this result and requiring symmetry of the combination (132) reveals that the integrability condition imposes the following six partial differential equations on the potential $\mathfrak{V}$:

$$\begin{aligned}
\mathfrak{V}_2 + 8p_4\mathfrak{V}_{24} &= -\mathfrak{V}_{11} - 4p_3\mathfrak{V}_{14}, \\
\tfrac{3}{2}\mathfrak{V}_3 + 12p_4\mathfrak{V}_{34} &= -2\mathfrak{V}_{12} + 4p_2\mathfrak{V}_{14}, \\
2\mathfrak{V}_4 + 16p_4\mathfrak{V}_{44} &= -3\mathfrak{V}_{13} - 4p_1\mathfrak{V}_{14}, \\
-3\mathfrak{V}_{13} - 12p_3\mathfrak{V}_{34} &= -4\mathfrak{V}_{22} + 8p_2\mathfrak{V}_{24}, \\
-4\mathfrak{V}_{14} - 16p_3\mathfrak{V}_{44} &= -6\mathfrak{V}_{23} - 8p_1\mathfrak{V}_{24}, \\
-8\mathfrak{V}_{24} + 16p_2\mathfrak{V}_{44} &= -9\mathfrak{V}_{33} - 12p_1\mathfrak{V}_{34}.
\end{aligned} \qquad (135)$$

Solutions of these equations that are polynomials in $\mathfrak{X}_m$ can be found by construction, and they are conveniently classified according to the power q of $\mathfrak{X}_1$ appearing in the polynomial. With some calculation, we have established that the unique polynomial solutions for $q \leq 4$ are

$$\begin{aligned}
\mathfrak{Y}_0 &= 1, \\
\mathfrak{Y}_1 &= \tfrac{1}{2}\mathfrak{X}_1, \\
\mathfrak{Y}_2 &= \tfrac{1}{8}(\mathfrak{X}_1^2 - 2\mathfrak{X}_2), \\
\mathfrak{Y}_3 &= \tfrac{1}{48}(\mathfrak{X}_1^3 - 6\mathfrak{X}_1\mathfrak{X}_2 + 8\mathfrak{X}_3), \\
\mathfrak{Y}_4 &= \tfrac{1}{384}(\mathfrak{X}_1^4 - 12\mathfrak{X}_1^2\mathfrak{X}_2 + 12\mathfrak{X}_2^2 + 32\mathfrak{X}_1\mathfrak{X}_3 - 48\mathfrak{X}_4).
\end{aligned} \qquad (136)$$

More generally, it follows that any polynomial obtained as the term at $O(\mathfrak{C}^q)$ in the series expansion of $\sqrt{|\det[1+\mathfrak{C}\psi]|}$ is a solution. An expression for these polynomials is

$$\mathfrak{Y}_q = \lim_{\epsilon\to 0}\frac{1}{q!}\frac{\partial^q}{\partial\epsilon^q}(\epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \epsilon^4 p_4)^{1/2}. \qquad (137)$$

For example, at $q = 5$ a solution to Eq. (135) is the polynomial

$$\begin{aligned}
\mathfrak{Y}_5 &= \frac{1}{768}(-3\mathfrak{X}_1^5 + 28\mathfrak{X}_1^3\mathfrak{X}_2 - 36\mathfrak{X}_1\mathfrak{X}_2^2 \\
&\quad - 48\mathfrak{X}_1^2\mathfrak{X}_3 + 32\mathfrak{X}_2\mathfrak{X}_3 + 48\mathfrak{X}_1\mathfrak{X}_4). \qquad (138)
\end{aligned}$$

We conjecture that the polynomials obtained in this way are in fact unique solutions at each order q.

A general potential $\mathfrak{V}$ that solves the differential equations (135) can therefore be written as

$$\sqrt{|\psi|}\mathfrak{V} = \sqrt{|\psi|}\sum_{q=0}^{\infty}\alpha_q\mathfrak{Y}_q, \qquad (139)$$

where the α_q are arbitrary real constants. For any fixed α_q, a potential of this form satisfies the integrability conditions (128) required for the bootstrap procedure. Note that for the special case $\alpha_q = \alpha_0$ for all $q \geq 0$, the solution becomes

$$\sqrt{|\psi|}\mathfrak{V} = \alpha_0\sqrt{|\det[\psi + \mathfrak{C}]|}. \qquad (140)$$

D. Bootstrap for integrable potential

In this subsection, we first apply the bootstrap procedure to the integrable potential (139). We then consider some aspects of extrema of the resulting theory, provide a construction for a local minimum, and offer some remarks about alternative bootstrap procedures for the potential.

1. Potential bootstrap

The bootstrap procedure using the cardinal field $\mathfrak{C}^{\mu\nu}$ can be explicitly performed term by term on the potential (139). For each q, $\sqrt{|\psi|}\mathfrak{Y}_q$ is a coefficient in the expansion of $\sqrt{|\det[\psi+\mathfrak{C}]|}$. In Sec. III B 2, a bootstrap procedure has been performed that leads to the potential (85) proportional to $\sqrt{|\det[\psi+\mathfrak{C}]|}$. It follows from this analysis that the bootstrap applied to the term $\sqrt{|\psi|}\mathfrak{Y}_q$ generates for each q the full result $\sqrt{|\det[\psi + \mathfrak{C}]|}$ minus the sum of all terms of orders less than q:

$$\begin{aligned}
&\sqrt{|\psi|}\mathfrak{Y}_0 \to \sqrt{|\det[\psi + \mathfrak{C}]|},\\
&\sqrt{|\psi|}\mathfrak{Y}_1 \to \sqrt{|\det[\psi + \mathfrak{C}]|} - \sqrt{|\psi|}\mathfrak{Y}_0,\\
&\sqrt{|\psi|}\mathfrak{Y}_2 \to \sqrt{|\det[\psi + \mathfrak{C}]|} - \sqrt{|\psi|}(\mathfrak{Y}_0 + \mathfrak{Y}_1),
\end{aligned} \tag{141}$$

and so on, with the general term being

$$\sqrt{|\psi|}\mathfrak{Y}_q \to \sqrt{|\det[\psi + \mathfrak{C}]|} - \sqrt{|\psi|}\sum_{k=0}^{q-1}\mathfrak{Y}_k. \tag{142}$$

Applying the bootstrap to the general potential (139) yields the bootstrap potential $\mathfrak{V}_\mathfrak{C}$,

$$\begin{aligned}
\sqrt{|\psi|}\mathfrak{V}_\mathfrak{C} &= \sum_{q=0}^{\infty}\alpha_q\Big(\sqrt{|\det[\psi + \mathfrak{C}]|} - \sqrt{|\psi|}\sum_{k=0}^{q-1}\mathfrak{Y}_k\Big)\\
&= \sqrt{|\psi|}\sum_{q=0}^{\infty}\alpha_q\sum_{k=q}^{\infty}\mathfrak{Y}_k = \sqrt{|\psi|}\sum_{k=0}^{\infty}\delta_k\mathfrak{Y}_k,
\end{aligned} \tag{143}$$

where the real coefficients δ_k are given as

$$\delta_k = \sum_{q=0}^{k}\alpha_q. \tag{144}$$

Note that the coefficient δ_k for fixed k acquires nonvanishing contributions from any nonvanishing coefficients α_q with $q \le k$.

For nonlinear cardinal gravity, the above discussion reveals that the potential term appearing in the bootstrap action takes the form

$$S_{\mathfrak{V},\mathfrak{C}} = \int d^4x\mathfrak{V}_\mathfrak{C} = \sum_{k=0}^{\infty}\delta_k\int d^4x\mathfrak{Y}_k. \tag{145}$$

This potential term combines with the kinetic term $S_{\mathfrak{K},\mathfrak{C}}$ in Eq. (111) to form the primary cardinal action.

2. Extrema of the potential

Vacuum solutions of nonlinear cardinal gravity are extremal solutions of the potential $\mathfrak{V}_\mathfrak{C}$. In an extremum, the cardinal field $\mathfrak{C}^{\mu\nu}$ acquires a vacuum value that may differ from any extrema generated by the potential $\mathfrak{V}$ in the linearized theory and defined in Eq. (98). By mild abuse of notation, in what follows we adopt the same notation $\mathfrak{C}^{\mu\nu} = \mathfrak{c}^{\mu\nu}$ for a vacuum value in an extremum of $\mathfrak{V}_\mathfrak{C}$. Similarly, we adopt the same notation as in Eq. (104) for the decomposition of the cardinal field $\mathfrak{C}^{\mu\nu}$ and its fluctuations $\tilde{\mathfrak{C}}^{\mu\nu}$ into the NG excitations $\mathfrak{N}^{\mu\nu}$ of Eq. (105) and the massive excitations $\mathfrak{M}^{\mu\nu}$ of Eq. (106). However, linearized results for $\mathfrak{N}^{\mu\nu}$ and $\mathfrak{M}^{\mu\nu}$ such as Eqs. (107) and (108) no longer hold.

A vacuum of $\mathfrak{V}_\mathfrak{C}$ can also be identified by the values $\mathfrak{x}_m$ taken by the four scalars $\mathfrak{X}_m$, as in Eq. (122). The restriction of the potential $\mathfrak{V}_\mathfrak{C}$ to the NG sector can then be achieved by replacing $\mathfrak{V}_\mathfrak{C}$ with the Lagrange-multiplier potential

$$\mathfrak{V}_\lambda = \sum_{m=1}^{4}\lambda_m(\mathfrak{X}_m - \mathfrak{x}_m), \tag{146}$$

which excludes fluctuations away from the extremum. If desired, the on-shell values of the Lagrange multipliers λ_m can be set to zero by suitable boundary conditions. This potential facilitates the identification of the NG and massive modes. The NG modes $\mathfrak{N}^{\mu\nu}$ are the nonzero components of $\mathfrak{C}^{\mu\nu}$ that preserve the constraints obtained from the Lagrange-multiplier equations of motion, while the massive modes are the components of $\mathfrak{C}^{\mu\nu}$ that are constrained to zero. Note that the potential $\mathfrak{V}_\lambda$ is dynamically equivalent to a potential $\mathfrak{V}_{\lambda'}$ expressed using the integrable polynomials (137), given by

$$\mathfrak{V}_{\lambda'} = \sum_{m=1}^{4}\lambda'_m(\mathfrak{Y}_m - \mathfrak{y}_m), \tag{147}$$

where $\mathfrak{y}_m$ are the values of $\mathfrak{Y}_m$ for $\mathfrak{C}^{\mu\nu} = \mathfrak{c}^{\mu\nu}$. The Lagrange-multiplier constraints are equivalent by direct comparison, while the dynamical properties under variation with respect to $\mathfrak{C}^{\mu\nu}$ are equivalent when the Lagrange multipliers are identified by the nonsingular set of linear equations

$$\lambda_m = \frac{(-1)^{m+1}}{2m}\sum_{p=m}^{4}\lambda'_p\mathfrak{y}_{p-m} \tag{148}$$

with $1 \le m \le 4$.

Using the potential (146), the NG modes $\mathfrak{N}^{\mu\nu}$ are seen directly to be the solutions of the equations $\mathfrak{X}_m = \mathfrak{x}_m$, which can be written as nonlinear generalizations of Eq. (107),

$$\begin{aligned}
0 &= \mathrm{tr}[\mathfrak{N}\eta],\\
0 &= 2\,\mathrm{tr}[\mathfrak{N}\eta c\eta] + \mathrm{tr}[(\mathfrak{N}\eta)^2],\\
0 &= 3\,\mathrm{tr}[\mathfrak{N}\eta(c\eta)^2] + 3\,\mathrm{tr}[(\mathfrak{N}\eta)^2 c\eta] + \mathrm{tr}[(\mathfrak{N}\eta)^3],\\
0 &= 4\,\mathrm{tr}[\mathfrak{N}\eta(c\eta)^3] + 3\,\mathrm{tr}[(\mathfrak{N}\eta)^2(c\eta)^2] + 3\,\mathrm{tr}[(\mathfrak{N}\eta c\eta)^2]\\
&\quad + 4\,\mathrm{tr}[(\mathfrak{N}\eta)^3 c\eta] + \mathrm{tr}[(\mathfrak{N}\eta)^4].
\end{aligned} \tag{149}$$

The ten independent components of $\mathfrak{N}^{\mu\nu}$ are constrained by these four equations, leaving the expected six NG modes. The four massive modes can be denoted by $\mathfrak{M}_m$ and specified as

$$\mathfrak{M}_m = \mathfrak{X}_m - \mathfrak{x}_m = \mathrm{tr}[(\mathfrak{c}\eta + \tilde{\mathfrak{C}}\eta)^m] - \mathrm{tr}[(\mathfrak{c}\eta)^m]. \tag{150}$$

They are contained in the symmetric tensor $\mathfrak{M}^{\mu\nu}$, which is obtained by subtraction of the NG modes $\mathfrak{N}^{\mu\nu}$ from the cardinal fluctuation field $\tilde{\mathfrak{C}} = \mathfrak{C}^{\mu\nu} - \mathfrak{c}^{\mu\nu}$.

In the absence of coupling to matter, the equations of motion for cardinal gravity are obtained by varying the sum of the kinetic and potential actions (111) and (145) with respect to the independent fields. Eliminating the auxiliary field $\Gamma^\alpha{}_{\mu\nu}$ yields the field equations in the absence of matter as

$$R_{\mu\nu} = 2\kappa\tau^{\rm vac}_{\mu\nu}, \qquad \mathfrak{X}_m = \mathfrak{x}_m, \tag{151}$$

where $\tau^{\rm vac}_{\mu\nu}$ is given by

$$\begin{aligned} -\frac{1}{2}\tau^{\rm vac}_{\mu\nu} &= \frac{\partial\mathfrak{V}_{\mathfrak{C}}}{\partial\mathfrak{C}^{\mu\nu}}\bigg|_{\mathfrak{C}\to\mathfrak{c}} = \sum_{m=1}^{4}\frac{\partial\mathfrak{X}_m}{\partial\mathfrak{C}^{\mu\nu}}\mathfrak{V}_{\mathfrak{C},m}|_{\mathfrak{C}\to\mathfrak{c}} \\ &= \sum_{m=1}^{4} m[\eta(\mathfrak{c}\eta)^{m-1}]_{\mu\nu}\mathfrak{V}_{\mathfrak{C},m}|_{\mathfrak{C}\to\mathfrak{c}}. \end{aligned} \tag{152}$$

Note that $\mathfrak{V}_{\mathfrak{C},m} = \lambda_m$ in the Lagrange-multiplier limit. The quantity $\tau^{\rm vac}_{\mu\nu}$ represents a kind of vacuum energy-momentum tensor density. Trace-reversing yields the field equations for cardinal gravity in the absence of matter, which can be written in the form

$$G^{\mu\nu} = 2\kappa T^{\mu\nu}_{\rm vac}. \tag{153}$$

Here, $G^{\mu\nu}$ is the Einstein tensor for the metric obtained from the metric density $(\eta^{\mu\nu} + \mathfrak{C}^{\mu\nu})$, while the vacuum energy-momentum tensor $T^{\mu\nu}_{\rm vac}$ is obtained by the corresponding trace reversal of $\tau^{\rm vac}_{\mu\nu}$. The conservation law

$$D_\mu T^{\mu\nu}_{\rm vac} = 0 \tag{154}$$

follows by virtue of the Bianchi identities. This conservation remains true in the presence of matter couplings, provided the matter-sector energy-momentum tensor is independently conserved. If the Lagrange multipliers λ_m vanish, or more generally if $\mathfrak{V}_m$ vanishes, then the vacuum energy-momentum tensor is zero and the usual form of general relativity is recovered. Otherwise, there is a positive or negative contribution to the vacuum energy-momentum tensor. This may play a role in cosmology and the interpretation of dark energy.

In the pure NG sector with zero on-shell Lagrange-multiplier fields, the effective potential vanishes and nonlinear cardinal gravity reduces to the kinetic term (113). As already noted, this limit reproduces general relativity, with the identification $\mathfrak{N}^{\mu\nu} \leftrightarrow \mathfrak{h}^{\mu\nu}$ in Eq. (114). The Einstein-Hilbert action is recovered in a fixed gauge, the nonlinear cardinal gauge, which is defined by the four nonlinear gauge conditions

$$\begin{aligned} 0 &= \mathrm{tr}[\mathfrak{h}\eta], \\ 0 &= 2\,\mathrm{tr}[\mathfrak{h}\eta c\eta] + \mathrm{tr}[(\mathfrak{h}\eta)^2], \\ 0 &= 3\,\mathrm{tr}[\mathfrak{h}\eta(c\eta)^2] + 3\,\mathrm{tr}[(\mathfrak{h}\eta)^2 c\eta] + \mathrm{tr}[(\mathfrak{h}\eta)^3], \\ 0 &= 4\,\mathrm{tr}[\mathfrak{h}\eta(c\eta)^3] + 3\,\mathrm{tr}[(\mathfrak{h}\eta)^2(c\eta)^2] + 3\,\mathrm{tr}[(\mathfrak{h}\eta c\eta)^2] \\ &\quad + 4\,\mathrm{tr}[(\mathfrak{h}\eta)^3 c\eta] + \mathrm{tr}[(\mathfrak{h}\eta)^4] \end{aligned} \tag{155}$$

obtained by the replacement $\mathfrak{N}^{\mu\nu} \to \mathfrak{h}^{\mu\nu}$ in Eq. (149).

The bootstrap for general relativity transforms the gauge symmetry (37) of the linearized theory into diffeomorphism invariance of the Einstein-Hilbert action, involving particle transformations of the metric density $\mathfrak{g}^{\mu\nu}$. In the linear cardinal theory, the analogue of the gauge symmetry (37) is the symmetry (5) of the kinetic term alone. The prebootstrap potential $\mathfrak{V}$ explicitly breaks this symmetry, so the potential term (145) can be expected to exhibit diffeomorphism breaking under particle transformations of the analogue metric density $((\eta^{\mu\nu} + \mathfrak{C}^{\mu\nu}))$. This is reflected, for example, in the presence of a factor $\sqrt{|\psi|} \to \sqrt{|\eta|} = 1$ in the measure of Eq. (145). However, as expected from the match to general relativity, the pure NG sector of cardinal gravity with zero on-shell Lagrange multipliers does exhibit the usual diffeomorphism invariance because the potential vanishes in this sector. Note also that cardinal gravity remains invariant under diffeomorphisms of the Minkowski spacetime.

Both general relativity and cardinal gravity are invariant under (observer) general coordinate transformations. The match between the two theories in the pure NG limit involves a coordinate transformation taking $(\eta^{\mu\nu} + \mathfrak{c}^{\mu\nu}) \to \eta^{\mu\nu}$ in the kinetic term (113). There is therefore a corresponding transformation taking $\eta^{\mu\nu} \to [(1 + \mathfrak{c}\eta)^{-1}\eta]^{\mu\nu}$ in the potential term. For example, the general coordinate invariance ensures a factor $\sqrt{|(1 + \mathfrak{c}\eta)^{-1}\eta|}$ appears in the measure of Eq. (145). However, the vanishing of the potential in the pure NG sector makes this factor irrelevant for the match to general relativity.

3. Stability of the extrema

Given a bootstrap potential $\mathfrak{V}_{\mathfrak{C}}$, an interesting issue is whether it admits an extremum that is stable. The question of overall stability for any given theory with Lorentz violation is involved [23]. Even for the comparatively simple bumblebee theories the issue remains open, although considerable recent progress has been made [24]. A full analysis for cardinal gravity lies outside the scope of this work. Instead, this subsection provides a few remarks on stability in the specific context of the potential term.

In the vacuum, the extremal solutions obey

$$0 = \frac{\partial\mathfrak{V}_{\mathfrak{C}}}{\partial\mathfrak{C}^{\mu\nu}}\bigg|_{\mathfrak{C}\to\mathfrak{c}} = \sum_{m=1}^{4} m[\eta(\mathfrak{c}\eta)^{m-1}]_{\mu\nu}V_{\mathfrak{C},m}|_{\mathfrak{C}\to\mathfrak{c}}, \tag{156}$$

where $\mathfrak{V}_{\mathfrak{C},m} \equiv \partial\mathfrak{V}_{\mathfrak{C}}/\partial\mathfrak{X}_m$. By assumption, the matrix $\mathfrak{c}\eta$ has four inequivalent nonzero eigenvalues. Working in the basis in which $\mathfrak{c}\eta$ is diagonal, this implies the generic conditions for a vacuum are

$$\mathfrak{V}_{\mathfrak{C},m}|_{\mathfrak{C}\to\mathfrak{c}} = 0. \tag{157}$$

A vacuum of $\mathfrak{V}_{\mathfrak{C}}$ is stable if it is a Morse critical point with positive definite hessian. For simplicity, we introduce the explicit diagonal basis

$$\mathfrak{C}^{\mu\lambda}\eta_{\lambda\nu} = \mathfrak{C}_\mu \delta^\mu{}_\nu \tag{158}$$

(no sum on μ), where the four quantities $\mathfrak{C}_\mu$ are the eigenvalues of $\mathfrak{C}\eta$. Then

$$\mathfrak{X}_m = \sum_{j=0}^{3} (\mathfrak{C}_j)^m, \tag{159}$$

and in the vacuum $\mathfrak{C}_j = \mathfrak{c}_j$, with all four values $\mathfrak{c}_j$ inequivalent and nonzero. In the diagonal basis, stability depends on the hessian

$$H_{jk} = \frac{\partial^2\mathfrak{V}_{\mathfrak{C}}}{\partial\mathfrak{C}_j\partial\mathfrak{C}_k}\bigg|_{\mathfrak{C}\to\mathfrak{c}} = \sum_{m,n=1}^{4} mn(\mathfrak{c}_j)^{m-1}(\mathfrak{c}_k)^{n-1}\mathfrak{V}_{\mathfrak{C},mn}|_{\mathfrak{C}\to\mathfrak{c}}. \tag{160}$$

If the discriminant is nonzero and the four eigenvalues H_m of the hessian are positive, the extremum is a local minimum.

An analytical derivation of a potential with a positive definite hessian in terms of the polynomial basis (137) is challenging. Instead, we proceed by ansatz using the shifted variables

$$\tilde{\mathfrak{X}}_m = \mathfrak{X}_m - \mathfrak{x}_m. \tag{161}$$

For the ansatz, we adopt the form of a Taylor expansion

$$\mathfrak{V}_{\mathfrak{C}} = \tfrac{1}{2}a_{mn}\tilde{\mathfrak{X}}_m\tilde{\mathfrak{X}}_n + \tfrac{1}{6}a_{mnp}\tilde{\mathfrak{X}}_m\tilde{\mathfrak{X}}_n\tilde{\mathfrak{X}}_p + \ldots, \tag{162}$$

where the coefficients $a_{mn}, a_{mnp}, \ldots$ are real constants. The potential $\mathfrak{V}_{\mathfrak{C}}$ in Eq. (145) is a combination of integrable partial potentials, so the expression (162) must be integrable too. We can therefore constrain the coefficients by imposing the integrability conditions (135) on $\mathfrak{V}_{\mathfrak{C}}$ itself at $\tilde{\mathfrak{X}}_m = 0$. At second order in $\tilde{\mathfrak{X}}_m$, this imposes six conditions on the ten degrees of freedom a_{mn}. The four degrees of freedom a_{m4} can be taken as unconstrained at this order. To impose the integrability conditions at third order, it is convenient to take partial derivatives of Eqs. (135) with respect to each $\mathfrak{X}_m$. This produces 24 equations, which combine with the second-order equations to yield 16 independent constraints on the 20 third-order coefficients a_{mnp}. The four degrees of freedom a_{m44} can be taken as unconstrained at this order. Proceeding in this way, we find a $4(n-1)$-dimensional solution space for the potential $\mathfrak{V}_{\mathfrak{C}}$ up to order n. As a check, the resulting solutions can be reconstructed in terms of suitable combinations of the polynomial basis (137).

Given the potential $\mathfrak{V}_{\mathfrak{C}}$ in the form (162), the issue of finding a solution with positive definite hessian can be resolved numerically. Investigation shows that there is a subspace of coefficients a_{mn} for which the integrability conditions are satisfied and the hessian is positive definite. An explicit example is the potential

$$\mathfrak{V}_{\mathfrak{C}} = \sum_{k=1}^{8} \delta_k \mathfrak{Y}_k, \tag{163}$$

with the coefficients given by

$$\begin{aligned} &\delta_1 \simeq -2.81, &&\delta_2 \simeq -5.46, &&\delta_3 \simeq 13.1, &&\delta_4 \simeq 19.3,\\ &\delta_5 \simeq -24.7, &&\delta_6 \simeq -29.6, &&\delta_7 \simeq 16.0, &&\delta_8 \simeq 17.1. \end{aligned} \tag{164}$$

The local minimum is found to lie at

$$\mathfrak{X}_1 \simeq 0.250, \qquad \mathfrak{X}_2 \simeq 2.06, \qquad \mathfrak{X}_3 \simeq 0.578, \qquad \mathfrak{X}_4 \simeq 1.44. \tag{165}$$

The eigenvalues of the corresponding hessian are found to be

$$H_1 \simeq 2.80, \qquad H_2 \simeq 0.927, \qquad H_3 \simeq 0.104, \qquad H_4 \simeq 0.0579, \tag{166}$$

demonstrating positivity. This example therefore represents a potential $\mathfrak{V}_{\mathfrak{C}}$ having a local minimum.

4. Alternative potential bootstraps

The bootstrap procedure discussed above holds for the potential prior to the development of a vacuum value for the cardinal field $\mathfrak{C}^{\mu\nu}$. Alternative options for the potential term, applicable following spontaneous Lorentz violation instead, include a secondary bootstrap using the cardinal fluctuation $\tilde{\mathfrak{C}}^{\mu\nu}$ and a tertiary one using only the NG modes $\mathfrak{N}^{\mu\nu}$. The explicit construction of these potentials lies outside the scope of this work. Instead, this subsection contains a few brief comments about some aspects of these alternative bootstrap procedures, following from the analysis of the primary case.

To perform an alternative bootstrap procedure, the corresponding integrable potential must first be constructed. For the secondary bootstrap involving the cardinal fluctuation $\tilde{\mathfrak{C}}^{\mu\nu}$ introduced in Eq. (102), the promotion of the potential $\mathfrak{V}$ to a covariant expression with respect to the auxiliary metric density $\psi^{\alpha\beta}$ involves the four scalars $\mathfrak{X}_m$ given by

$$\mathfrak{X}_m(\psi) = \mathrm{tr}[(\mathfrak{c}\psi + \tilde{\mathfrak{C}}\psi)^m]. \tag{167}$$

The energy-momentum tensor must now be obtained from an action by varying with respect to $\tilde{\mathfrak{C}}^{\mu\nu}$. The basic integrability condition is found to be

$$\frac{\delta^2(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\mu\nu}\delta\tilde{\mathfrak{C}}^{\alpha\beta}} = \frac{\delta^2(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\alpha\beta}\delta\tilde{\mathfrak{C}}^{\mu\nu}}. \tag{168}$$

However, since the cardinal fluctuation $\tilde{\mathfrak{C}}^{\mu\nu}$ is merely a constant shift of the cardinal field $\mathfrak{C}^{\mu\nu}$, we have

$$\frac{\partial\mathfrak{X}_m}{\partial\tilde{\mathfrak{C}}^{\mu\nu}} = \frac{\partial\mathfrak{X}_m}{\partial\mathfrak{C}^{\mu\nu}}. \tag{169}$$

This in turn means that the integrability condition is satisfied for the same symmetry requirement on the same expression (132) as before. The integrable potential for the secondary bootstrap therefore takes the same form (139) as for the primary case.

A similar situation holds for the tertiary bootstrap involving the NG modes $\mathfrak{N}^{\mu\nu}$ in the decomposition (104). In this case, the four relevant scalars are

$$\mathfrak{X}_m = \mathrm{tr}[(\mathfrak{c}\psi + \mathfrak{N}\psi + \mathfrak{M}\psi)^m]. \tag{170}$$

The energy-momentum tensor is required to arise by varying an action with respect to $\mathfrak{N}^{\mu\nu}$. This generates the integrability condition

$$\frac{\delta^2(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\mu\nu}\delta\mathfrak{N}^{\alpha\beta}} = \frac{\delta^2(\sqrt{|\psi|}\mathfrak{V})}{\delta\psi^{\alpha\beta}\delta\mathfrak{N}^{\mu\nu}}. \tag{171}$$

However, the form of Eq. (104) implies

$$\frac{\partial\mathfrak{X}_m}{\partial\mathfrak{N}^{\mu\nu}} = \frac{\partial\mathfrak{X}_m}{\partial\mathfrak{C}^{\mu\nu}}. \tag{172}$$

It follows that the integrability condition is again satisfied for the same symmetry requirement on the same expression (132), and the integrable potential for the tertiary bootstrap takes the same form (139) as before.

Although the integrable potentials (139) are the same, the alternative bootstrap procedures differ from each other and from the primary one presented above. Moreover, performing these bootstrap procedures involves additional choices because integration with respect to the linear cardinal fluctuation or the linear NG modes can either be continued at all orders or can be adjusted at each order to incorporate the induced nonlinearities. Any of these bootstrap procedures could in principle be performed using the methods presented in Sec. III.

An extremum of an alternative bootstrap potential is achieved for vanishing massive modes. It can therefore be represented by a suitable Lagrange-multiplier potential. In particular, in the pure NG limit the potential vanishes for on-shell multipliers, and so the resulting effective theory is controlled by the corresponding kinetic term. This means that general relativity is also recovered in the low-energy limits of the nonlinear theories arising in these alternative bootstrap procedures.

V. COUPLING TO MATTER

At the linear level, the cardinal field $C^{\mu\nu}$ must couple to other fields in the Minkowski spacetime via a symmetric two-tensor current. Given our gravitational interpretation of the cardinal field, the other fields in the theory can be regarded as the matter. They provide one natural two-tensor current, the energy-momentum tensor $T_{\mathrm{M}\mu\nu}$ in the Minkowski spacetime. We can therefore expect the linearized theory (1) to incorporate the matter interaction

$$\mathcal{L}^{L}_{\mathrm{M},C} = \tfrac{1}{2}C^{\mu\nu}T_{\mathrm{M}\mu\nu}. \tag{173}$$

No coupling constant is necessary for this interaction, since it can be absorbed in the scaling factor κ already present in the original theory (1).

A. Primary bootstrap

The bootstrap procedure involving the cardinal field $\mathfrak{C}^{\mu\nu}$ can be applied to the matter interaction (173) to determine the form of the matter coupling for cardinal gravity. For this purpose, the interaction (173) is conveniently expressed in terms of the trace-reversed energy-momentum tensor $\tau_{\mathrm{M}\mu\nu}$ for the matter. This tensor arises by variation of the Lagrange density $\mathcal{L}_\mathrm{M}$ for the matter fields via

$$-\frac{1}{2}\tau_{\mathrm{M}\mu\nu} = \frac{\delta\mathcal{L}_\mathrm{M}(\eta\to\psi)}{\delta\psi^{\mu\nu}}\bigg|_{\psi\to\eta} \tag{174}$$

in the usual way. We can therefore write

$$\mathcal{L}^{L}_{\mathrm{M},C} = -\tfrac{1}{2}\mathfrak{C}^{\mu\nu}\tau_{\mathrm{M}\mu\nu} \tag{175}$$

for the matter interaction with the cardinal field $\mathfrak{C}^{\mu\nu}$.

To perform the bootstrap, the techniques of Sec. III B can be applied. The Lagrange density (175) is linear in $\mathfrak{C}^{\mu\nu}$ and so has the form (69), for which the bootstrap yields Eq. (75). The bootstrap therefore generates the Lagrange density

$$\mathcal{L}_{\mathrm{M},\mathfrak{C}} = \mathcal{L}_\mathrm{M}|_{\eta\to\eta+\mathfrak{C}}. \tag{176}$$

Some insight into the physical content of this result can be obtained by expanding about an extremum of the bootstrap potential. Writing $\mathfrak{C}^{\mu\nu} = \mathfrak{c}^{\mu\nu}$ in the extremum and denoting the corresponding fluctuations by $\tilde{\mathfrak{C}}^{\mu\nu} = \mathfrak{N}^{\mu\nu} + \mathfrak{M}^{\mu\nu}$ as before, we obtain

$$\mathcal{L}_{\mathrm{M},\mathfrak{C}} = \mathcal{L}_\mathrm{M}|_{\eta\to\eta+\mathfrak{c}+\tilde{\mathfrak{C}}}. \tag{177}$$

A comparison of this result to the matter coupling of general relativity can be performed by adopting the Lagrange-multiplier bootstrap potential (146). The massive modes vanish, $\mathfrak{M}^{\mu\nu}\to 0$, and as before a suitable change of coordinates must be performed to implement the transformation $(\eta^{\mu\nu} + \mathfrak{c}^{\mu\nu}) \to \eta^{\mu\nu}$ and thereby ensure the kinetic term (113) contains the conventional Minkowski metric. The resulting Lagrange density $\mathcal{L}^{\mathrm{NG}}_{\mathrm{M},\mathfrak{C}}$ then matches the usual matter term $\mathcal{L}^{\mathrm{GR}}_{\mathrm{M}}$ in general relativity,

$$\mathcal{L}^{\rm NG}_{\rm M,\mathfrak{C}} = \mathcal{L}_{\rm M}|_{\eta\to\eta+\mathfrak{N}} \leftrightarrow \mathcal{L}^{\rm GR}_{\rm M} = \mathcal{L}_{\rm M}|_{\eta\to\mathfrak{g}}, \qquad (178)$$

when the correspondence $\mathfrak{g}^{\mu\nu} \leftrightarrow \eta^{\mu\nu} + \mathfrak{N}^{\mu\nu}$ of Eq. (115) is adopted.

We can therefore conclude that the pure NG sector of cardinal gravity with zero on-shell Lagrange multipliers exactly reproduces general relativity, including the matter coupling. When the massive modes are included, the matter coupling deviates from that in general relativity by terms that are suppressed by the scale of the massive modes.

B. Alternative bootstraps

Alternative bootstrap procedures for the matter coupling can be countenanced instead. We consider here the secondary and tertiary procedures discussed above for the kinetic and potential terms. We also examine some experimental implications of the results for the pure NG sector and the match to general relativity.

The secondary bootstrap involving the cardinal fluctuation $\tilde{\mathfrak{C}}^{\mu\nu}$ starts from the matter coupling (173) in the form

$$\mathcal{L}^{L}_{\rm M,\tilde{\mathfrak{C}}} = \mathfrak{c}^{\mu\nu}(-\tfrac{1}{2}\tau_{{\rm M}\mu\nu}) + \tilde{\mathfrak{C}}^{\mu\nu}(-\tfrac{1}{2}\tau_{{\rm M}\mu\nu}). \qquad (179)$$

The bootstrap can be performed using the methods of Sec. III B. The first term in Eq. (179) involves $\mathfrak{c}^{\mu\nu}$ but is independent of $\tilde{\mathfrak{C}}^{\mu\nu}$, while the only dependence on the Minkowski metric appears in $\tau_{{\rm M}\mu\nu}$. The effect of the bootstrap on this term is therefore to replace $\tau_{{\rm M}\mu\nu}(\eta^{\mu\nu})$ with $\tau_{{\rm M}\mu\nu}(\eta^{\mu\nu}+\tilde{\mathfrak{C}}^{\mu\nu})$, introducing a suitable power of $\sqrt{|\eta+\tilde{\mathfrak{C}}|}$ as needed. The second term is linear in $\tilde{\mathfrak{C}}^{\mu\nu}$ and hence is of the form (69), for which the bootstrap gives Eq. (75). We therefore obtain

$$\mathcal{L}_{\rm M,\tilde{\mathfrak{C}}} = c^{\mu\nu}(-\tfrac{1}{2}\tau_{{\rm M}\mu\nu}|_{\eta\to\eta+\tilde{\mathfrak{C}}}) + \mathcal{L}_{\rm M}|_{\eta\to\eta+\tilde{\mathfrak{C}}} \qquad (180)$$

as the secondary bootstrap matter coupling.

For the tertiary bootstrap, the starting point is the matter coupling in the form

$$\mathcal{L}^{L}_{\rm M,\mathfrak{N},\mathfrak{M}} = (c^{\mu\nu}+\mathfrak{M}^{\mu\nu})(-\tfrac{1}{2}\tau_{{\rm M}\mu\nu}) + \mathfrak{N}^{\mu\nu}(-\tfrac{1}{2}\tau_{{\rm M}\mu\nu}). \qquad (181)$$

Here, we bootstrap only the field $\mathfrak{N}^{\mu\nu}$ containing the linearized NG modes, without correcting at each order. Using the techniques in Sec. III B, we find the Lagrange density

$$\mathcal{L}_{\rm M,\mathfrak{N},\mathfrak{M}} = (c^{\mu\nu}+\mathfrak{M}^{\mu\nu})(-\tfrac{1}{2}\tau_{{\rm M}\mu\nu}|_{\eta\to\eta+\mathfrak{N}}) + \mathcal{L}_{\rm M}|_{\eta\to\eta+\mathfrak{N}} \qquad (182)$$

as the result of the tertiary bootstrap.

The alternative results (180) and (182) for the matter coupling contain terms corresponding to the usual minimally coupled Lagrange density for matter and additional couplings between matter and the massive modes. Each also contains a term involving the cardinal vacuum value $\mathfrak{c}^{\mu\nu}$ and the energy-momentum tensor. This last term remains as an unconventional expression in the Lagrange density in the pure NG limit $\mathfrak{M}^{\mu\nu}\to 0$, and for the match to general relativity it therefore represents an unconventional contribution to the matter sector.

Couplings involving tensor vacuum values appear naturally in the standard-model extension (SME), which provides a general framework for the description of Lorentz violation using effective field theory [2,25]. The matter sector of the SME includes Lorentz-violating operators controlled by coefficients that are symmetric observer two-tensors and that can be related to $c^{\mu\nu}$. Numerous experimental measurements have been performed on the coefficients for Lorentz violation [26]. This offers an interesting opportunity to identify constrains on the alternative bootstrap theories.

Consider first an example illustrating the connection between the cardinal matter coupling and the SME framework, involving a matter Lagrange density for a complex scalar field ϕ in Minkowski spacetime given by

$$\mathcal{L}^{0}_{\phi} = -\eta^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi - U(\phi^\dagger\phi). \qquad (183)$$

Here, $U(\phi^\dagger\phi)$ is an effective Lorentz-invariant potential that can include mass and self-interaction terms. The corresponding energy-momentum tensor $T^0_{\mu\nu}$ is

$$T^0_{\mu\nu} = \partial_\mu\phi^\dagger\partial_\nu\phi + \partial_\nu\phi^\dagger\partial_\mu\phi + \eta_{\mu\nu}\mathcal{L}^0_\phi. \qquad (184)$$

Introducing the cardinal coupling (173) and restricting attention to the vacuum value $c^{\mu\nu}$ adds the term

$$\mathcal{L}^{\phi}_{c} = \tfrac{1}{2}c^{\mu\nu}T^0_{\mu\nu} = \mathfrak{c}^{\mu\nu}(-\tfrac{1}{2}\tau^0_{\mu\nu}), \qquad (185)$$

where $\tau^0_{\mu\nu}$ is the trace-reversed form of $T^0_{\mu\nu}$. Performing either of the alternative bootstraps in the NG limit yields the contribution of the cardinal-scalar coupling to the full theory,

$$\begin{aligned}\mathcal{L}^{\phi}_{c,\mathfrak{N}} &= \sqrt{|\mathfrak{g}|}[\mathfrak{c}^{\mu\nu}(-\tfrac{1}{2}\tau^0_{\mu\nu})]|_{\eta\to\mathfrak{g}} = \tfrac{1}{2}\sqrt{|g|}c^{\mu\nu}T^0_{\mu\nu}|_{\eta\to g}\\ &= \tfrac{1}{2}\sqrt{|g|}c^{{\rm T}\mu\nu}T^0_{\mu\nu}|_{\eta\to g} + \tfrac{1}{8}\sqrt{|g|}\,{\rm tr}[cg]\,{\rm tr}[T^0|_{\eta\to g}g],\end{aligned} \qquad (186)$$

where we denote the bootstrap metric density $\eta^{\mu\nu}+\mathfrak{N}^{\mu\nu}$ by $\mathfrak{g}^{\mu\nu}$ and the corresponding metric by $g_{\mu\nu}$. For the last expression in this equation, the coefficient $c^{\mu\nu}$ has been separated into traceless and trace pieces for convenience in what follows, via the definitions

$$c^{\mu\nu} = c^{{\rm T}\mu\nu} + \tfrac{1}{4}{\rm tr}[cg]g^{\mu\nu}, \qquad {\rm tr}[c^{\rm T}g] = 0. \qquad (187)$$

We can compare the result for $\mathcal{L}^{\phi}_{c,\mathfrak{N}}$ to that obtained in the SME framework for the Lorentz-violating theory of a complex scalar field in Riemann spacetime with Lagrange density [2]

$$\mathcal{L}_g^\phi = -\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi - \sqrt{|g|}U(\phi^\dagger\phi) + \tfrac{1}{2}\sqrt{|g|}k^{\mu\nu}(\partial_\mu\phi^\dagger\partial_\nu\phi + \partial_\nu\phi^\dagger\partial_\mu\phi). \quad (188)$$

In this model, $k^{\mu\nu}$ is a symmetric coefficient for Lorentz violation, which is normally taken to satisfy $\mathrm{tr}[kg] = 0$ because a nonzero trace is Lorentz invariant. Inspection reveals the identification

$$c^{\mathrm{T}\mu\nu} \equiv k^{\mu\nu} \quad (189)$$

between the cardinal vacuum value and the SME coefficient for Lorentz violation. Note that the conformally invariant case satisfies $\mathrm{tr}[T^0 g] = 0$, in which case the two models (186) and (188) match exactly.

As another example with direct physical application, consider the Maxwell Lagrange density $\mathcal{L}_{\mathrm{EM}}^{(0)}$ for photons in Minkowski spacetime, given in Eq. (77). The corresponding energy-momentum tensor $T_{\mu\nu}^{\mathrm{EM}}$ is presented in Eq. (76). The cardinal-photon coupling is

$$\mathcal{L}_c^\phi = \tfrac{1}{2}c^{\mu\nu}T_{\mu\nu}^{\mathrm{EM}} = \mathfrak{c}^{\mu\nu}(-\tfrac{1}{2}\tau_{\mu\nu}^{\mathrm{EM}}), \quad (190)$$

and the bootstrap generates the result

$$\mathcal{L}_{\mathfrak{c},\mathfrak{R}}^{\mathrm{EM}} = \sqrt{|\mathfrak{g}|}[\mathfrak{c}^{\mu\nu}(-\tfrac{1}{2}\tau_{\mu\nu}^{\mathrm{EM}})]|_{\eta\to g} = \tfrac{1}{2}\sqrt{|g|}c^{\mathrm{T}\mu\nu}F_\mu{}^\alpha F_{\nu\alpha}. \quad (191)$$

In this example, only the traceless part $c^{\mathrm{T}\mu\nu}$ appears in the final answer because the photon action is conformally invariant. This result can be compared to the CPT-even part of the photon sector in the minimal SME [27]. The corresponding coefficients for Lorentz violation form an observer four-tensor $(k_F)^{\alpha\lambda\mu\nu}$, which has the symmetries of the Riemann tensor. This four-tensor can be decomposed in parallel with the decomposition of the Riemann tensor into the Weyl tensor, the traceless Ricci tensor, and the scalar curvature. The scalar part is Lorentz invariant. The Weyl part involves an observer four-tensor that controls birefringence of light induced by Lorentz violation. The traceless Ricci part determines the anisotropies in the propagation of light due to Lorentz violation, and it is specified by the traceless observer two-tensor $k_F^{\mu\nu} \equiv (k_F)^\alpha{}_{\mu\alpha\nu}$. Only the latter effects are relevant for present purposes. Restricting attention to these coefficients produces in Riemann spacetime the Lagrange density [2]

$$\mathcal{L}_{\mathrm{EM}} = -\tfrac{1}{4}\sqrt{|g|}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\sqrt{|g|}k_F^{\mu\nu}F_\mu{}^\alpha F_{\alpha\nu} = -\tfrac{1}{4}\sqrt{|g|}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\sqrt{|g|}k_F^{\mu\nu}T_{\mu\nu}^{\mathrm{EM}}, \quad (192)$$

where the tracelessness of $k_F^{\mu\nu}$ has been used. Comparison of this result with Eq. (191) shows the match

$$c^{\mathrm{T}\mu\nu} \equiv k_F^{\mu\nu}, \quad (193)$$

in analogy with that of Eq. (189).

The similarity of the matches (189) and (193) between $c^{\mathrm{T}\mu\nu}$ and certain traceless SME coefficients for Lorentz violation is no accident. Consider a theory in which the spacetime metric in the gravity sector is $g_{\mu\nu}$. If the theory has Lorentz violation, the matter-sector metric could differ from $g_{\mu\nu}$. Denote the matter-sector metric by $g_{\mu\nu} + k_{\mu\nu}$, where the coefficient $k_{\mu\nu}$ for Lorentz violation is symmetric and traceless. For small $k_{\mu\nu}$, the matter-sector Lagrange density $\mathcal{L}_{\mathrm{M}}(g+k)$ can be expanded as

$$\mathcal{L}_{\mathrm{M}}(g+k) = \mathcal{L}_{\mathrm{M}}(g) + k_{\mu\nu}\frac{\delta\mathcal{L}_{\mathrm{M}}(g)}{\delta g^{\mu\nu}} + \ldots = \mathcal{L}_{\mathrm{M}}(g) + \frac{1}{2}k_{\mu\nu}T_{\mathrm{M}}^{\mu\nu} + \ldots, \quad (194)$$

where $T_{\mathrm{M}}^{\mu\nu}$ is the energy-momentum tensor for the Lagrange density $\mathcal{L}_{\mathrm{M}}(g)$. We see that the piece of the cardinal coupling (173) involving $c^{\mathrm{T}\mu\nu}$ can always be matched at leading order to a term involving a traceless shift $k_{\mu\nu}$ in the matter-sector metric of a theory with Lorentz violation.

The same line of reasoning also yields a path to experimental constraints on $c^{\mathrm{T}\mu\nu}$. The key point is that a suitable choice of coordinates can convert $g_{\mu\nu} + k_{\mu\nu} \to g'_{\mu\nu}$, thereby making the matter sector Lorentz invariant at leading order in $k_{\mu\nu}$. The price for this transformation is the conversion of the gravity-sector metric $g_{\mu\nu} \to g'_{\mu\nu} - k_{\mu\nu}$, which means that signals from Lorentz violation could be detectable in suitable gravitational experiments. In particular, at leading order we find

$$\mathcal{L}_{\mathrm{cardinal}} \supset \kappa\mathfrak{g}^{\mu\nu}R_{\mu\nu}(\Gamma) \to \kappa\mathfrak{g}'^{\mu\nu}R_{\mu\nu}(\Gamma) + \kappa k^{\mu\nu}R_{\mu\nu}(\Gamma). \quad (195)$$

The last term matches the standard form for one type of Lorentz violation in the gravity sector of the minimal SME, controlled by the coefficient $s^{\mu\nu}$ for Lorentz violation [2]. This coefficient can be studied experimentally in various ways [28,29]. Most components of related coefficients have been constrained to parts in 10^5 to 10^{10} via reanalysis of several decades of data from lunar laser ranging [30] and by laboratory tests with atom interferometry [31]. We can therefore conclude that the traceless part of the vacuum value of the cardinal field is constrained at the same level in both the secondary and the tertiary cardinal theories.

VI. SUMMARY AND DISCUSSION

This work constructs an alternative theory of gravity, which we call cardinal gravity, based on the idea that gravitons are massless NG modes originating in spontaneous Lorentz violation. The starting point is the simple theory (1) of a symmetric two-tensor cardinal field $C^{\mu\nu}$ in Minkowski spacetime with a potential triggering spontaneous Lorentz violation [6]. Requiring consistent self-coupling to the energy-momentum tensor constrains the form of the potential to the form (139). It also defines a bootstrap procedure that permits the construction of a self-consistent nonlinear theory.

When the bootstrap is applied to the original theory prior to the spontaneous Lorentz violation, cardinal gravity emerges. This theory has kinetic term $S_{\mathfrak{K},\mathfrak{C}}$ given by Eq. (111), potential term $S_{\mathfrak{V},\mathfrak{C}}$ given by Eq. (145), and matter coupling $\mathcal{L}_{\mathrm{M},\mathfrak{C}}$ given by Eq. (176). At low energies compared to the scale of the massive modes, the potential can be approximated by its extremal Lagrange-multiplier form (146) that allows only NG excitations about the vacuum. In this limit, the nonlinear cardinal action reduces to the Einstein-Hilbert action of general relativity with conventional matter coupling and possibly a vacuum energy-momentum term (152), all expressed in the nonlinear cardinal gauge given by Eq. (155).

If instead the bootstrap is applied to the effective action for the spontaneously broken theory, alternative cardinal theories are generated. Using the fluctuation field about the cardinal vacuum value as the basis for the bootstrap yields a secondary cardinal gravity. This has kinetic term given by Eq. (116) and matter coupling given by Eq. (180). Using instead only the NG excitations to perform the bootstrap produces a tertiary cardinal gravity, with kinetic term given by Eq. (119) and matter coupling given by Eq. (182). The actions of these alternative cardinal theories also reduce to the Einstein-Hilbert action in the pure NG limit and in the nonlinear cardinal gauge (155). However, unconventional matter coupling terms remain in this limit. These can be constrained by suitable gravitational experiments, and existing results limit the magnitude of components of the cardinal vacuum value to parts in 10^5 to 10^{10}.

All forms of cardinal gravity differ from general relativity in certain respects. One is the presence of the massive modes $\mathfrak{M}^{\mu\nu}$. The scale of these modes is set by the curvature of the potential about the Lorentz-violating extremum. The natural scale in the theory is the Planck mass, which enters via the Newton gravitational constant in the usual way, so it is plausible that the fluctuations of the modes $\mathfrak{M}^{\mu\nu}$ are also of Planck mass. At low energies, their propagation can therefore be neglected, and they can be integrated out of the action to yield their effective contribution. The form of the kinetic term (111) suggests the corrections to the Einstein-Hilbert action appear in part as the square of the Ricci tensor suppressed by the square of the mass of the modes $\mathfrak{M}^{\mu\nu}$. A suppressed effective matter self-interaction that is quadratic in the energy-momentum tensor also appears. Investigation of the resulting subleading corrections to the Einstein equations, some of which are proportional to the Ricci tensor and hence vanish in the vacuum, is an open topic. A post-Newtonian study of the experimental consequences for laboratory and solar-system situations, including gravitational-wave searches, would be of definite interest. A study of the implications for cosmology would also be worthwhile because corrections appear to standard solutions and also because the vacuum energy-momentum tensor (152) can appear. These various investigations may be most effectively undertaken in the nonlinear cardinal gauge (155), for which the form of conventional general-relativistic solutions remains to be obtained.

In more extreme situations, such as near the singularities of black holes or in the very early Universe, the contributions from the massive modes could be sufficient to change qualitatively the usual general-relativistic behavior. The additional propagating modes can be expected to affect features such as inflation and to change the cosmic gravitational background. At sufficiently high temperatures the potential changes shape [32] to restore exact Lorentz symmetry, with an extremum having a zero value for $C_{\mu\nu}$. This reverse phase transition converts the NG modes into massive modes, so the graviton excitations acquire Planck masses and the nature of gravity at the big bang is radically changed.

Cardinal gravity has general coordinate invariance and diffeomorphism symmetry of the background spacetime at all scales, as discussed in the context of the gauge-fixing conditions (155). Diffeomorphism invariance involving the analogue metric density $(\eta^{\mu\nu} + \mathfrak{N}^{\mu\nu})$ emerges in the low-energy limit, where the match to general relativity occurs. This feature of cardinal gravity has some appeal. The aesthetic and mathematical advantages of the diffeomorphism invariance of general relativity are maintained in the low-energy limit of cardinal gravity, while at high energies the presence of the original background spacetime may offer conceptual and calculational advantages for understanding the physics. One example might involve the vacuum value of the metric, which is presumably set by processes at the Planck scale. In general relativity one can ask why the vacuum value of the metric is nonzero. Since the metric is the fundamental field and the Einstein-Hilbert action has diffeomorphism invariance, it might seem natural for the metric field to vanish in the vacuum. In contrast, in cardinal gravity at high energies the background spacetime is nondynamical, and the gravitational properties at high energies are controlled instead by the cardinal field. The vacuum value of the cardinal field affects the physics but not the existence of spacetime properties. Another example might be improved prospects for quantum calculations at high energies, although this would require revisiting the analysis in the present work with quantum physics in mind. For instance, our derivation of the integrable potential is based on purely classical considerations, and the effect of radiative corrections is an open issue. In the context of bumblebee theories, requiring one-loop stability under the renormalization group restricts the form of the potential and shows that those producing spontaneous Lorentz breaking are generic [33]. The analogue of this for cardinal gravity represents an independent condition on the potential that is likely to constrain further its form.

We conclude this discussion by noting an interesting possibility implied by the present work. We have demonstrated here that nonlinear gravitons in general relativity can be interpreted as NG modes from spontaneous Lorentz violation. It is also known that photons can be interpreted as NG modes from spontaneous Lorentz violation, even in the presence of gravity: the Einstein-Maxwell equations are reproduced at low energies by a suitable bumblebee theory [5]. Both the graviton and the photon have two physical propagating modes. However, spontaneous Lorentz violation and the accompanying diffeomorphism violation can generate up to ten NG modes [5], so the possibility exists in principle of developing a combined cardinal-bumblebee theory in which the graviton and the photon simultaneously emerge as NG modes from spontaneous Lorentz violation. This would represent an alternative unified framework for understanding the long-range forces in nature.

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